Florian Cajori

PLANE GEOMETRY

DEVELOPED BY THE

SYLLABUS METHOD

BY

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W. P. I

PREFACE

THE belief that the proofs of Geometry should be, as far as possible, worked out by the pupils, either in class discussions or individually, is becoming more widespread every year. The day of memorizing proofs will soon be past, and the most efficient method for mental training along logical lines will be the one generally adopted. This syllabus is written with the hope of encouraging teachers to undertake Geometry by the "no text" method. The author believes very decidedly that this method gives a maximum of mental training with a minimum waste of energy.

The list of theorems is based on the latest reports of the Mathematical Associations, and, while much shorter than that in many of the text-books, it will be found sufficient to prepare the pupils for any of the colleges. It contains all the theorems of the "New England List" with a few additions that simplify proofs.

The order is the development of ten years' class use, and will be found different from that of any text. Whenever a theorem has seemed to be simplified, either in content or in proof, by making a change in its place in the order of theorems, that change has been tried in class, and has been made permanent if it proved of advantage. Any teacher using this book should feel equally free to make changes in the order if he is convinced that there is a decided advantage in the change.

The chapter on Logic has been found of great assistance in helping the pupils to think accurately, and it is certain to save more time for a class than its discussion requires.

The definitions and axioms are given in quite complete form, not for assignment to the class, as this part of the work should be developed *before* the text is given to a class for study; but as a guide for the teacher, both in order and in subject-matter, and as a reference book for the pupil. Good results can be secured by withholding the book from the pupils until part, at least, of the preliminary matter has been discussed.

The subjects of "existence" and "betweenness" have not been considered to any great extent, as they do not seem worth the time and effort required, except to a student of the more advanced pure mathematics. "Location" and "intersection," on the other hand, are of such vital importance in considering the correctness of proofs that they have received some attention. The aim throughout has been to arrange a system of Geometry that should be natural, reasonably complete, and suitable to afford as much mental training as the maturity of the pupils would allow. The author has not hesitated to assume any axiom that would help more than its presence would complicate; on the other hand, he has left out things that seemed to require more than they gave.

Geometry itself has no concern with measurement by means of a unit. The applications of Geometry to such measurement are, however, very frequent and very important, and while this book presupposes geometrical proofs to as great a degree as seems possible without unnecessarily complicating the subject, there has been no attempt to draw a hard and fast line of demarcation between Geometry and its applications. If a teacher believes in distinguishing sharply between the different branches of mathematics, the study of the lengths of line sects and the calculation of areas can be put under the head of MENSURATION.

The exercises are in two divisions, those under the theorems and those in general lists. The exercises under the theorems have been chosen to illustrate the uses of the various theorems, and they should therefore be of great help to the teacher. The general lists give the pupils practice in finding for themselves what principles underlie the proofs. Probably no class could finish all the exercises in the book in one year, but the teacher can easily choose those best suited to his purposes. There are several pages of college examination questions. Some of these are duplicates of exercises scattered through the book, but the differences in wording, as well as the desire to let students know what type of questions examiners ask, has prompted leaving them in the book.

This book has been written with little reference to the order and methods of other texts, for it is a compilation that has grown naturally from class work. The author is, of course, indebted for many of the ideas used to numerous works on mathematics and its pedagogy, but in many cases it is now impossible to tell from what source the suggestion first arose. He wishes, however, to acknowledge his special indebtedness to Dr. William H. Metzler of Syracuse University, for assistance and encouragement in the writing of this book.

EUGENE R. SMITH.

SUGGESTIONS TO TEACHERS

Preliminary Definitions and Axioms. Do not assign these to the class to read until after they have been thoroughly discussed in class. As far as possible, let the pupils frame the definitions for themselves, and lead up naturally to the simple deductions from them (starred in the text), so that the pupils can begin to discover these truths from the very beginning.

Every word of this part need not be digested before going on to the theorems; but before any new work is undertaken the teacher should make sure that the pupils have a perfect grasp of the particular facts to be used in developing the new matter. Certain parts can be touched upon lightly on the first reading, and cleared up thoroughly just before being used. *Make haste slowly.*

Theorems. Develop the very difficult ones in class by the question and answer method of analysis; assign those less difficult for outside preparation or to be worked out in class, either at the board or at the pupils' seats. Clear up the work frequently by review recitations covering the theorems recently done. Use a great deal of oral work, asking for every viewpoint on both new and old work. *Never tell the pupils what to do; ask questions so framed that the pupils are made to think.* Summarize frequently, and by the use of exercises cultivate the pupils' power to choose the correct method.

Notebooks. The author advises that the pupils be required to keep notebooks in which they write out all the theorems, unless the conditions under which the teacher is working make it impossible. All necessary corrections can be noted by the teacher and made by the pupil, after which the theorems can be filed, and so become at the end of the course a complete reference book compiled by the pupil. A loose-leaf system will be found convenient.

Time Required. *Do not hurry the early work.* Whatever time is used in obtaining a thorough understanding of the foundations and of the methods will be more than repaid by increased speed later on.

The first quarter of the year's course should be spent on the work up to the section on Locus in Book I, and if this assignment is not quite finished, no great harm will be done.

The second quarter should almost, if not quite, finish the second book.

The third quarter should take about to the constructions of the fourth book.

The last quarter should finish the syllabus and leave at least a month for general review and additional exercise work.

This is an estimate of a fair average rather than of an excellent record, and many classes can do much better; it supposes that a good deal of original work on exercises, and almost daily oral work on uses of theorems, methods of attack, and other important topics have been taken.

Logic. Make frequent use of the section on Logic. Ask the pupils to state the converse and the contraposite of very many of the theorems, and to discover whether these are new theorems. In this way the proofs of many theorems and of many excellent exercises will be found.

Limits and the Incommensurable Case. In common with many of the best educators, the author believes that the proofs of these theorems are too difficult for immature minds. He therefore recommends that the Appendix proofs be omitted, but that some explanation of the possibility of the incommensurable case be given.

Methods of Attack. It is the experience of the author that the methods of pure logic and the classification and elimination method are the ones of principal importance for proving theorems. The so-called "Indirect Method" and "Reductio ad Absurdum" are unnecessary if logic is understood. "Intersection of Loci," while following directly from the classification method, has its uses, and the analysis method for constructions is invaluable. The analyses of the sample theorems will serve as a guide to the class method of discovering the proof for a theorem.

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SUMMARY OF GEOMETRICAL SIGNS

$+$ plus, sign of addition
$-$ minus, sign of subtraction
$\times$ times, sign of multiplication
$\div$, /, :, divided by, sign of division
$\sqrt{}$ square root sign
$=$ is (or are) equal, or equivalent
$\neq$ is not equal, or equivalent, to
$\equiv$ is identical to
$\cong$ is congruent to
$\doteq$ approaches as a limit
$\sim$ is similar to
$>$ is greater than
$\ngtr$ is not greater than
$<$ is less than
$\nless$ is not less than
$\parallel$ is parallel to
$\perp$ is perpendicular to
$\angle$ or $\measuredangle$ angle
$\triangle$ triangle
▱ parallelogram
▭ rectangle
□ square
$\odot$ circle
$\frown$ arc
$\therefore$ therefore
$\because$ because, since

The signs for figures become plural by the addition of *s*, often within the sign, as ▱s for rectangles.

OTHER ABBREVIATIONS

eq. + eq., for "equals plus equals," and similarly for the other axioms.

2 s. incl. $\angle$, for "two sides and the included angle," etc.

In referring to propositions or axioms as authority, it is permissible to use any easily understood abbreviation for the theorem. The ones given above are good examples of abbreviations that, if used correctly in a theorem, could hardly be misunderstood.

PLANE GEOMETRY

BOOK I. RECTILINEAR FIGURES

SECTION I. LOGIC

1. Need of Logic. In all discussions, more or less logic is used, though often unknowingly. For example, if a person, noticing that a flag is hanging limp on a pole, says, "There is no wind to-day," that person has reasoned as follows:

"If the wind were blowing, the flag would be waving. But the flag is not waving, so the wind is not blowing."

Because such reasoning is so instinctive that we seldom realize the steps through which the mind arrives at the conclusion, the need of a knowledge of the most important laws of logic is not fully appreciated.

In Geometry, the same kind of reasoning must be applied, and as instinct with regard to the figures to be studied is not reliable, a brief study of the most commonly applied laws of logic is necessary.

2. Conditional Statements. Practically all statements are conditional, though the condition is sometimes implied rather than stated. Even the statement, "It rains," depends upon the place and time.

When a statement asserts that the truth of one thing assures the truth of another thing, the one upon which the other depends is the *condition* (or *hypothesis*), the other the *conclusion* of the statement.

EXAMPLES:

CONDITION	CONCLUSION
If it rains to-day,	I will stay at home.
If a triangle has two equal sides,	the opposite angles are equal.
A horse	has four legs.
If John goes down town,	I will go with him.

NOTE. In the second example, the word "triangle" simply indicates the figure in which the reasoning is to be, and for the purposes of Geometry it does not need to be considered as part of either the condition or the conclusion. The terms used in this illustration have not been defined as yet, but they are common enough to be familiar to most students.

3. Four Related Statements. There are four related conditional statements which can be formed from two possible truths used in the positive and the negative. Let the two possible truths be "it rains," and "the sidewalk is wet"; then the following related statements might be made (with no consideration at present as to whether they are true statements or not):

(1) If it rains, the sidewalk will be wet.
(2) If the sidewalk is wet, it has rained.
(3) If it does not rain, the sidewalk will not be wet.
(4) If the sidewalk is not wet, it has not rained.

If the two possible facts were represented by A and B, where A and B stand for any two possibilities such that one might depend upon the other, these statements could be written as follows:

(1) If A, then B.
(2) If B, then A.
(3) If not A, then not B.
(4) If not B, then not A.

4. Converse. The first and second statements have the condition of each the same as the conclusion of the other; or, *the condition and conclusion of either interchanged will form the other.* They are called *converse* to each other.

5. Negative or Obverse. The first and third have the condition and conclusion of either the negative of the condition and the conclusion of the other; or, *the condition and conclusion of either made negative will form the other.* They are called *negative* or *obverse* to each other.

6. Negative Converse or Contraposite. The first and fourth have the condition of each the negative of the conclusion of the other; or, *the condition and the conclusion of either interchanged and made negative will form the other.* They are called *negative converse* or *contraposite* to each other. The reason for the name "negative converse" is evident from the definitions of these three relations in §§ 4, 5, and 6.

7. Exercises.

1. Write three conditional statements, separate each into condition and conclusion, and write the converse, negative, and negative converse of each.

2. What relation have statements 2 and 3? 2 and 4? 3 and 4?

3. What relation to the original statement has
(*a*) the converse of its negative?
(*b*) the converse of its negative converse?
(*c*) the negative of its negative converse?
(*d*) the negative of the converse of its negative converse?

8. Contraposite (or Negative Converse) Law. *The contraposite of any true conditional statement is also true.*

Note. This law is discussed more fully in the Appendix, § 339.

It is true also that the negative converse of any false statement is false, although this is of less importance than the former statement.

The most common way of using this method of reasoning is to make some true statement that a certain conclusion follows a certain condition; then, on finding that the conclusion is not true, to say that the condition is not true. The statement about the flag in § 1 was an example of this. Another example of a very self-evident kind is: One is looking for a book with a green cover, and finding a book the cover of which is not green, knows at once that the book is not the book wanted. Definitely put, the argument would be:

The book wanted has a green cover.

This book has not a green cover, so it is not the right book.

Contraposite reasoning is probably the most common kind of reasoning, for almost every one uses it many times daily. It is, however, used more or less unconsciously in daily life, while in Mathematics and Science, and in many other places where the reasoning needs to be perfectly accurate, one must know definitely just what steps are being taken, in order that no error can creep in.

Exercises. Suppose that if A is true, B is also true; what other statement regarding A and B is known to follow? Write five true statements, then see if their contraposites are also true. See if their converses are true.

9. Truth of the Negative and the Converse of a True Statement. The negative and the converse of a single true statement are not *necessarily* true. They are either both true, or both false, for they are negative converse to each other. The discussion of when they are true will be taken up in § 10. The following examples will show that they are not always true:

"An apple tree has leaves." It does not follow that

(1) If it is not an apple tree, it has not leaves.

(2) If it has leaves, it is an apple tree.

"This desk is made of wood." It does not follow that

(1) If it is not this desk, it is not made of wood.

(2) If it is made of wood, it is this desk.

"A donkey has a head." It does not follow that because you have a head you are a donkey.

WARNING. *Never assume that the negative or the converse of ONE true statement is also true.*

10. Law of Converse. *If conditional statements such that their conditions cover all possibilities, and no two conclusions can be true at once, are true, then the converses of those statements are also true.*

NOTE. This law is discussed more fully in the Appendix, § 340.

This law will be understood when it is applied to definite cases, but the following examples make its meaning somewhat clearer.

(1) If A is true, X is true.

If A is not true, X is not true.

These two conditions (true and untrue) cover all possibilities, and the two conclusions (true and untrue) are such that they cannot both be true at the same time, so the converses are also true; *i.e.*

If X is true, A is true.

If X is not true, A is not true.

(2) If $A > B$, $X > Y$.

If $A = B$, $X = Y$.

If $A < B$, $X < Y$.

These three conditions ($>$, $=$, $<$) cover all possibilities, and but one of the conclusions ($>, =, <$) can be true at once, so the converses are also true; *i.e.*

$$\text{If } X > Y,\ A > B.$$
$$\text{If } X = Y,\ A = B.$$
$$\text{If } X < Y,\ A > B.$$

11. General and Special Cases.

NOTE. This need not be read until the pupil is ready to begin the theorems.

Anything known of mankind is known of each man separately; but anything known of certain men only, would not necessarily be true of all men. If it is true for each man in existence, it is true for all mankind. In other words, anything known of a class as a whole, or of all members of the class, is known to be a characteristic of that class, both as a whole and by individuals. On the other hand, anything known of part of a class is not known of the class as a whole, or of other members of that class.

So in Geometry, proofs should be made for the general case whenever that is possible, and when that does not seem possible, the proof should be worked for each of the different cases separately. For example, in working with triangles, the triangle used should always be one about which no assumption (other than the given of the theorem) is made. The triangle should neither be isosceles, nor be assumed to have any certain sized angle, such as a right or acute angle. It is better to draw the triangle in the figure so that it does not even *appear* to have any special characteristic, or the one studying the figure may carelessly assume that the characteristic which the figure appears to have really belongs to it.

In theorems where it does not seem possible to find a proof for the general case (and these are comparatively rare), it is necessary to prove enough cases to cover all possibilities in order that the general case may be known; for example, a proof for right, acute, and obtuse-angled triangles would be true for all triangles. In the same way, a proof for triangles having three equal sides, two equal sides, and no equal sides, would be true for all triangles.

Sometimes a proof is true only for a special case on account of points or lines that are added to the figure and are assumed to lie in certain positions, when as a matter of fact they can equally well lie in other positions. If points or lines are added to a figure, the proof must hold for all possible positions in which they can lie. This is discussed in more detail in § 110.

SECTION II. POINTS, LINES, AND SURFACES

12. Geometry. This subject studies points, lines, and figures formed by them. It proves facts about the figures, and uses as a basis for the reasoning *definitions* and *axioms.*

13. Definitions. A *definition* is such a description of the thing defined as will distinguish it from all other things; it might be said to be an agreement as to what a term shall be used to indicate. Some things are of such simple nature that it is difficult, if not impossible, to define them in terms still simpler, and in such cases an explanation in regard to them may well take the place of a definition.

14. Axioms. A truth that is taken as one of the foundation facts of a subject is called an *axiom.* It is often defined as a truth so simple that it cannot be derived from truths still simpler; but for Elementary Geometry this is not strictly true (Appendix, § 343). It will be found that the axioms of Geometry are facts so self-evident that there is no doubt as to their truth.

15. Space. The space in which everything exists is, as far as experience shows, unlimited. At any rate, the space studied in Elementary Geometry (sometimes called Euclidean Space) is unlimited. Space is evidently divisible, for all bodies occupy portions of space.

16. Solids. Any limited portion of space, such as the space occupied by any body, is called — irrespective of the nature of the body which may occupy it — a *geometric* *solid*, or simply a *solid*. Solids are said to have three dimensions: *length*, *breadth*, and *thickness*.

17. Surfaces. That which separates one portion of space from an adjoining portion is called a *surface*. If two adjoining lots are considered as extending down into the ground so that any distance down there is still a boundary between the lots, that boundary is a surface. It evidently does not occupy space, for any particle of soil belongs to one lot or to the other, yet there is a distinct boundary such that all on one side of it belongs to one lot, while all on the other side of it belongs to the other lot.

NOTE. The terms "side," "between," "within," "outside," will be used in this syllabus in their ordinary meaning without any attempt to define them geometrically.

Another example of a surface is the outside of any object, as a box. It separates the space occupied by the object from the space outside the object, but itself occupies no space.

A surface may be limited or unlimited in extent, and may have limited portions; it is said to be two dimensional, having length and breadth.

18. Lines. That which separates one portion of a surface from an adjoining portion is called a *line*. The surface boundary between two lots is a line. The line occupies no space, yet definitely divides one lot from the other.

A line may be indefinite in extent, but has limited portions; it is said to be one dimensional, having length only.

19. Points. That which separates one portion of a line from an adjoining portion is called a *point*. If four lots come together in what is ordinarily called a corner, that corner is a point. It is a place, for it is fixed in position, but it occupies no space, and no portion of the surface. There is evidently no unowned spot at the corner, for all four lots extend to the point, leaving no unoccupied surface.

A point has no dimensions and is indivisible; it has position only.

20. These definitions might have been taken in reverse order; that is, starting with the point. A moving point passes through (or describes) a line; a moving line usually describes a surface; a moving surface usually describes a solid. It can be seen that it might be possible for a line to move along itself in such a way as not to describe a surface, and for a surface to fail to describe a solid.

21. Representations. Points, lines, and surfaces are represented by various things. The corner of a lot may be marked by a fence post, and the line by a fence, but it is evident that the post and the fence are not the point and the line. So in Geometry, a pencil or chalk mark may be used to represent a point or a line, but it is understood that they are used only to represent the things, *and that they are not the points and lines themselves.*

22. Straight Lines. The straight line is the most important and the most familiar kind of line. It is not easy to define, but some discussion of it is necessary. Different straight lines, or parts of the same straight line, need have but two points in common to coincide throughout, irrespective of how the lines are placed in other respects, and this characteristic of straight lines is the foundation of the definition.

The two following are probably the best definitions of the straight line:

(*a*) A line such that any part will, however placed, lie wholly on any other part, if its extremities are made to fall upon that other part, is called a straight line.

(*b*) A line that is determined by any two of its points is a straight line.

The word "determined" here means that the line is distinguished from any other line of the same kind by the fact that it goes through these particular two points; or, in other words, that it is the *only* straight line through the two points.

23. However the definition is worded, the fundamental fact about straight lines is the following:

Straight-line Axiom. *Through two points but one straight line can pass.*

An illustration to show that this distinguishes the straight line from all others can be made by folding a sheet of paper smoothly and drawing any line through two points on the edge formed, inside the fold. If the line is drawn in ink, and the sheet is folded firmly together, a second line exactly like the first, but in the opposite position, will appear from the trace of the ink, *unless the line is drawn along the edge*, in which case no second line will appear. The edge represents the straight line through the two points, and is the only one.

24. A second fact about straight lines which is based directly on the straight-line axiom is

* *Two different straight lines can intersect in but one point.*

Whenever a statement is marked with the mark *, that statement requires proof, the proof in the first part of the Geometry being always closely associated with the definitions and axioms. These proofs are sometimes given in the form of explanations, but the pupil should in all such cases be able to explain why the statement is true.

This is the first example of contraposite reasoning in the Geometry. It can be done as follows:

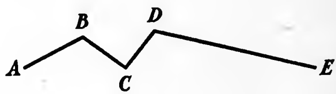
If the two lines met in a second point, there would be two different straight lines through two points.

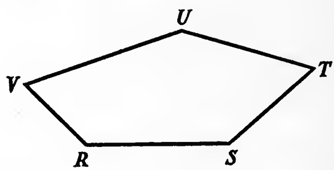
There cannot be two different straight lines through two points, by the straight-line axiom; therefore, by contraposite argument (*i.e.* the conclusion being untrue, the condition is also untrue), the two straight lines do not meet in two points.

In places where no ambiguity results, "line" will be used to mean straight line, since when any other kind of line is meant, the kind is always stated.

25. Sects. A limited portion of a line is called a *sect*, or a *line segment*. That part of a line that lies between the points A and B is a sect, and is called AB. Where there is nothing to indicate otherwise, sect will mean straight-line sect.

26. Broken Lines. A line composed of sects of different lines is called a *broken line*, as $ABCDE$. A broken line is said to be *closed* if it is continuous; that is, if the line, when traced from any point through its entire length, is found to return to the starting point. $ABCDE$ is not a closed line, but $RSTUV$ is closed.





The straight-line axiom shows that a straight line is not continuous, and that it does not intersect itself.

27. Closed-line Intersection Axiom. *If a straight line of indefinite length passes through a point within the surface inclosed by any closed line, it intersects the closed line at least twice.*

28. Curved Lines. A line, no part of which is straight, is called a *curved* line.

A curved line also is called closed if it is continuous. (See § 26.)

29. Geometrical Figures. Any combination of points, lines, and surfaces, formed under given conditions, is a *geometrical figure;* as, a *triangle* might be defined as the figure formed by three lines meeting in pairs. If the figure is formed by straight lines, it is called a *rectilinear figure.*

30. Planes. A surface in which, any two points being taken, the straight line which joins them lies wholly in the surface is called a *plane surface*, or simply a *plane*.

A plane might be defined as the surface determined by any three of its points that are not in the same straight line.

The plane occupies the same position among surfaces that the straight line holds among lines. As indicated by the first definition given, it is *straight* through any two of its points. The surface of a blackboard is a familiar example of a plane.

31. Plane Geometry. Plane Geometry treats only of geometric figures that lie entirely in the same plane.

SECTION III. EQUALITY

32. There are three words used in Geometry to denote equality: *congruent*, *equivalent*, and *equal*.

33. Congruence. Two figures are *congruent* when they can be made to coincide in every point.

34. Equivalence. Closed figures (or figures bounded by closed lines, see §§ 26, 28) are said to be *equivalent* when their boundaries inclose the same amount of surface.

35. Equality. The word *equal* is used somewhat in both senses, but in this syllabus it will be used only in those places where there can be no confusion between the ideas of congruence and equivalence. For example, sects will be said to be equal when they can be made to coincide, even though this fulfills the definition of congruence, for since a sect cannot inclose surface, there can be no confusion with equivalence.

36. Congruence includes equivalence, whereas equivalence does not imply congruence; a figure inclosed by a curved line might be equivalent to a figure inclosed by a broken line although it would be impossible to make them coincide.

37. Geometric Equality. In the strictest geometric sense, equality means that coincidence is possible, and in this the test for geometric equality differs from the test for arithmetic equality, for two arithmetic magnitudes are equal if they contain the same unit the same number of times. Evidently, then, equivalence is to some extent an arithmetic property, but Geometry is applied so often to calculations of magnitudes in terms of a unit, that it is neither necessary nor desirable to attempt to distinguish too carefully between it and other mathematical subjects. In practical work, Geometry will be found to have many parts that involve Arithmetic and Algebra, and while the distinctions between the subjects may be kept in mind, their combined use is entirely legitimate.

38. Equality Axioms.

(1) *Things equal to the same thing, or to equal things, are equal to each other.*

This axiom applies to both congruence and to equivalence; the first four following apply only to sects, angles (§ 44), and equivalent closed figures; they cannot be applied to the congruence of closed figures.

(2) *If equals are added to the same thing or to equal things, the results are equal.*

(3) *If equals are taken from the same thing or from equal things, the results are equal.*

(4) *If equals are multiplied by the same thing or by equal things, the results are equal.*

(5) *If equals are divided by the same thing or by equal things, the results are equal.*

(6) *The whole equals the sum of all its parts.*

39. Substitution Axiom. *A magnitude may be put in place of an equal magnitude in any equation or statement of inequality.*

This axiom is not independent of the equality axioms, but it is more convenient for many purposes. For example, if $ak-b=r$, and $k=l$; then, substituting l for k, $al-b=r$.

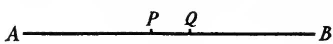
40. General and Geometric Axioms. Certain of the axioms apply not only to geometric magnitudes, but to all magnitudes, and are therefore called general axioms. The six equality axioms, the substitution axiom, and the axiom of inequality (§ 82), are general axioms. The straight-line axiom, and all others that refer to geometric conceptions only, are geometric axioms.

41. Axiom of Motion. *Geometric figures can be moved about in space without altering them in any way.* (Sometimes stated, "*without altering their size or shape.*") This axiom is used in testing congruence, for one figure is sometimes supposed to have been placed upon (or *superimposed upon*) another figure, and whatever is known about the two figures is then used to determine whether or not they coincide.

42. Axiom of Division. *Any magnitude can be divided into any number of equal parts.* (The number must be a positive integer.)

If the magnitude is divided into two equal parts, it is said to be *bisected;* if into three equal parts, to be *trisected*, and the parts are called *halves*, and *thirds*, respectively.

* **43.** *A sect can be bisected by but one point.*



If P and Q were both points of bisection (or *midpoints*) of AB, then AP and AQ would be equal, since each is $AB \div 2$ (eq. $\div$ eq.). Therefore AP and AQ coincide, and P falls on Q, making but one bisection point.

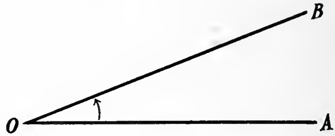
It might be thought at first that this fact was self-evident, but many things can be bisected in different ways, and it is important to know which ones have but one possible bisection. An apple, for instance, could be cut into halves in many different places.

The method of proof used to show that a sect has but one midpoint can also be used to show that a sect has but one point which cuts off one third of the sect from a certain end, and but one point which cuts off two thirds from that end. The method can also be extended to cover all cases of the division of a sect into equal parts.

NOTE. A method of proof can often be used for various purposes with very little change except for minor details. It has already been said that the proof in § 43 can be used for other divisions besides bisection. It is also true that it can be used for the bisection of an angle as well as of a sect, and for any other divisions of an angle. (See § 64.) It is very important that a student should become familiar with the method of work for a proof, rather than to attempt to memorize its relatively unimportant details. If the method is thoroughly understood, the details of the proof are not likely to prove troublesome.

SECTION IV. ANGLES

44. Angles. If two lines meet in a point, they are said to form an *angle*. The common point is called the *vertex*, and the lines the *arms* of the angle. The size of the angle is measured by the amount of rotation through which a line would pass in going from the position of one arm to the position of the other arm.



In the accompanying figure, the lines AO and BO form an angle, the size of the angle being measured by the amount of rotation necessary to pass from the position AO to the position BO.

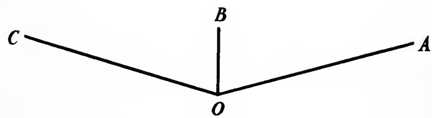
Since there are two ways in which to rotate from one line to the other, two angles are formed by two lines diverging from a point; when it is necessary to distinguish between the two, the letters on the arms are named in the order of direction contrary to that taken by the hands of a clock, the vertex letter being between the other two. The angle marked by the arrow (which shows the correct direction) would be named AOB, while the other angle would be BOA. Ordinarily the smaller of the two angles is meant, such an angle as BOA seldom being considered in Geometry.

If there is little chance of confusion as to which angle at a certain vertex is being used, the vertex letter alone is often used to name the angle; as, angle O.

It should be noticed that the size of an angle in no way depends on the length of its arms. One angle would be greater than, equal to, or less than, another, according as it would include, coincide with, or fall within, that other, if it were placed upon it with the vertex and one arm coinciding. An angle both of whose arms lie between the arms of a second angle is evidently smaller.

45. Sign for "Angle." The sign commonly used for "angle" is $\angle$; as $\angle AOB$. If there is a chance for confusion with the signs for "greater" and "less" ($>$ and $<$), a line is sometimes drawn across it, as $\measuredangle$. The commonest signs and abbreviations will be found following the index of terms.

46. Adjacent Angles. If two angles have a common vertex and lie on opposite sides of a common arm, they are called *adjacent* angles.

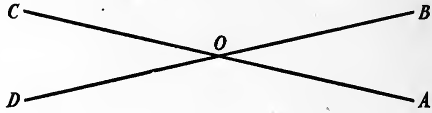


$\angle AOB$ and $\angle BOC$ are adjacent angles. They are adjacent whether or not the arms AO and OC lie in the same straight line.

47. Sum and Difference of Angles. The sum of two angles is the angle obtained by placing the angles adjacent to each other, and ignoring the common arm. The sum of $\angle AOB$ and $\angle BOC$ is $\angle AOC$ (in § 46).

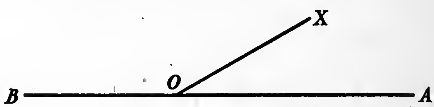
The difference of two angles is the angle obtained by placing the angles so that they have a common vertex, and lie on the *same side* of the common arm. The difference between $\angle AOC$ and $\angle BOC$ is $\angle AOB$ (in § 46).

48. Vertical Angles. If two lines intersect, forming four angles, any two angles that are not adjacent to each other are called *opposite*, or *vertical* angles.



$\angle AOB$ and $\angle COD$ are vertical, as are $\angle BOC$ and $\angle DOA$. The arms of one of two vertical angles are the arms of the other angle extended through the vertex.

49. Straight Angles. If the arms of an angle lie in the same straight line, but on opposite sides of the vertex, the angle is called a *straight* angle.



$\angle AOB$ is a straight angle. The sum of all the successive angles around a point, on one side of a straight line, is equal to a straight angle; for example, $\angle AOX + \angle XOB =$ st. $\angle$.

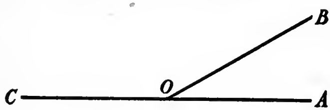
***50.** *All straight angles are equal.*

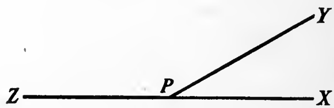
For the arms of one straight angle can be made to coincide with the arms of any other straight angle, since they form, in each case, a straight line, and so need have but two points in common to coincide throughout.

51. Supplements. If the sum of two angles is a straight angle, the angles are called *supplements* of each other.

In § 49 $\angle AOX$ and $\angle XOB$ are supplements.

***52.** *Supplements of the same angle, or of equal angles, are equal to each other.*





For if equal parts are taken from straight angles (which are equal), equal parts must be left, by equality axiom 3.

In the figure:
$$\text{st. } \angle AOC = \text{st. } \angle XPZ.$$
$$\text{If } \angle AOB = \angle XPY,$$
$$\text{then } \angle BOC = \angle YPZ. \text{ (eq.} - \text{eq.).}$$

NOTE. When one wishes to refer to some authority, as in this case to the equality axiom, any clear abbreviation may be used. The reference (eq. – eq.) means "equals minus equals," and refers to equality axiom 3. It is not best to refer to authorities by number; all authorities should be quoted in such a way that one could readily understand the meaning of the reference, and words or simple abbreviations are usually used.

***53**. *Any two vertical angles are equal.*

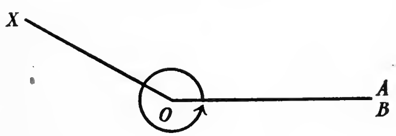
For they are supplements of the same angle.

In the figure of § 48 $\angle AOB$ and $\angle COD$ are both supplements of $\angle BOC$ (or of $\angle DOA$); similarly, $\angle BOC$ and $\angle DOA$ are both supplements of $\angle AOB$ (or of $\angle COD$).

54. Perigons. If the arms of an angle lie in the same straight line and on the same side of the vertex, one arm having rotated through two straight angles, the angle is called a *perigon*.

$\angle AOB$ is a perigon, since OB is supposed to have rotated around O (as indicated by the arrow) to the position OB on OA.

A perigon is the sum of the successive angles around a point; and it equals two straight angles. It is in the sense of a sum rather than as a single angle that perigon is most frequently used in Geometry. $\angle AOX + \angle XOB =$ a perigon.



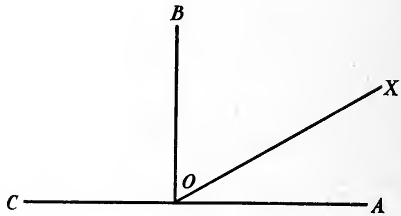
***55.** *All perigons are equal.*

For all straight angles are equal, and since a perigon is twice a straight angle, perigons are equal (eq. × eq.).

56. Explements. If the sum of two angles is a perigon, the angles are called *explements* of each other. In the figure of § 54 $\angle AOX$ and $\angle XOB$ are explements of each other.

***57.** *Explements of the same angle or of equal angles are equal.*

Notice the similarity to § 52.



58. Right Angles. If one line meets another line so as to make the two adjacent angles equal, the angles are called *right angles*, and the lines are said to be *perpendicular* to each other. If $\angle AOB = \angle BOC$, then these angles are right angles. A line not perpendicular to a second line is said to be an *oblique* to that line.

***59.** *A right angle is one half a straight angle.*

For by the definition of right angle, a straight angle is the sum of two equal right angles.

***60.** *All right angles are equal.*

For they are halves of equal straight angles. What axiom is used?

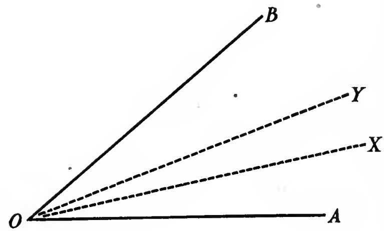
61. Complements. If the sum of two angles is a right angle, the angles are called *complements* of each other. In § 58 $\angle AOX$ and $\angle XOB$ are complements of each other.

*62. *Complements of the same angle or of equal angles are equal.*

See §§ 52 and 57.

63. Acute, Obtuse, Reflex Angles. An angle less than a right angle is called *acute;* one greater than a right angle, but less than a straight angle, is called *obtuse;* one more than a straight angle, but less than a perigon, is called *reflex.*

*64. *An angle can be bisected by but one line.*

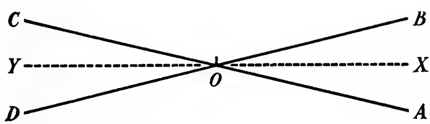


If OX and OY both bisect $\angle\ AOB$, then $\angle AOX = \angle AOY$ (each $= \angle\ AOB \div 2$, and eq. $\div$ eq.). Therefore the angles coincide, and OX falls on OY; that is, there is but one line which bisects the angle.

*65. *At a given point in a given line there can be but one perpendicular in the same plane.*

For the perpendicular bisects the straight angle having its vertex at its foot (§ 59), and there can be but one bisector of an angle (§ 64).

*66. *The bisectors of two vertical angles lie in one straight line.*

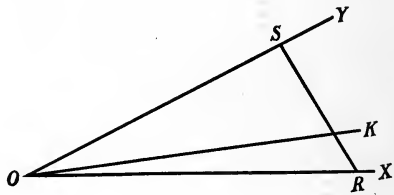


$\angle XOY$ is one half the perigon, or a straight angle, for the $\angle$s XOB, BOC, COY, are respectively equal to the $\angle$s AOX, DOA, YOD. Therefore $\angle XOY$ is a straight angle, and XY is a straight line.

*67. *The bisectors of two adjacent supplements are perpendicular to each other.*

For the angle between them is one half the straight angle.

68. Axiom of Intersection. *An indefinite line drawn from the vertex of an angle that is less than a straight angle, between the arms of the angle, intersects any line that joins a point on one arm of the angle with a point on the other arm of the angle.*



For example, the line OK meets RS, or any other line that joins any point on OX with a point on OY.

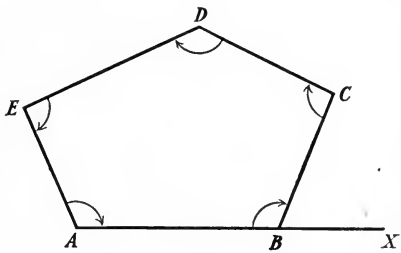
A consequence of this axiom that is needed for some of the proofs will be found in the Appendix, § 343.

SECTION V. POLYGONS

69. Polygons. The limited portion of a plane bounded by a broken line which is closed (§ 26) is called a *polygon.* The amount of surface within the polygon is called the *area of the polygon,* and the length of the boundary is called the *perimeter* of the polygon. The term "perimeter" is often used for the broken line bounding the polygon when no idea of length is involved, but when this is done, the context usually makes clear which meaning is to be attached to the word.

70. Vertices and Sides. The vertices of the angles formed by the sects of the broken line are called the *vertices* of the polygon, and the sects themselves are called the *sides* of the polygon. The figure *ABCDE* is a polygon.

71. Angles of a Polygon. The angles on the left in passing around a polygon in the direction contrary to that taken by the hands of a clock are called **interior angles.** The angles of *ABCDE* which are marked with the arrows are interior angles.



The angle at any vertex of a polygon having as arms one side of the polygon, and the continuation of another side, is called an **exterior angle,** as $\angle XBC$. In speaking of *the exterior angles of a polygon*, the angles formed by producing the sides in succession, each through the vertex formed with the following one, in passing around the polygon, are meant. There are evidently two sets of exterior angles of a polygon, formed by passing around clockwise or counterclockwise. These sets are, however, equal, for the two exterior angles at any vertex are vertical.

72. Diagonals. A line joining two non-consecutive vertices of a polygon is called a *diagonal.*

73. Concave, Convex, and Cross Polygons. A polygon is said to be *convex* if no side when produced could cut the surface of the polygon. Unless otherwise stated, convex polygon will be meant whenever the term "polygon" is used.

A polygon is *concave* when at least one side, if produced, would cut its surface; it is called *cross* when its perimeter intersects itself.

74. Equilateral, Equiangular, and Regular Polygons. A polygon that has all its sides equal is called *equilateral;* one that has all its angles equal is called *equiangular.* A polygon that is both equilateral and equiangular is called *regular.*

75. Number of Sides. A polygon of three, four, five, six, eight, ten sides is called, respectively, a *triangle*, *quadrilateral*, *pentagon*, *hexagon*, *octagon*, *decagon*, etc.

76. Base. The side of a polygon on which it appears to stand is called its *base.* Any side of a polygon might be considered the base.

TRIANGLES

77. Parts of a Triangle. A triangle has six *parts*, three sides and three angles. An angle and a side are spoken of as opposite to each other when the side is not one of the arms of the angle. A side is sometimes spoken of as *included* between two angles, and an angle as included between two sides, when the order in which they lie is meant.

78. Vertex Angle. The angle opposite the base of a triangle is called the *vertex* angle, or the *vertical* angle.

79. Equality of Sides. If a triangle has two equal sides, it is called *isosceles;* if no equal sides, *scalene.* In an isosceles triangle, the equal sides are sometimes called *legs*, the third side the *base.*

80. Angles of a Triangle. If all the angles of a triangle are acute, it is called an *acute-angled triangle;* if one angle is right, it is called a *right triangle;* and if one angle is obtuse, it is called an *obtuse-angled triangle.* In a right triangle the side opposite the right angle is called the *hypotenuse*, and the other sides the *legs.*

81. Lines of a Triangle. There are four kinds of lines of importance in work with a triangle: the *bisectors of the angles*, the *perpendicular bisectors of the sides*, the *altitudes*, and the *medians.* The first two explain themselves; an *altitude* is a perpendicular from a vertex to the opposite side, and a *median* is a line from a vertex to the midpoint of the opposite side. If the *altitude* of a triangle is spoken of, the altitude to the base is meant.

SECTION VI. INEQUALITIES

82. Axiom of Unequals. *The whole is greater than any of its parts.*

83. Inequalities. There are certain truths relating to statements of inequality that depend very closely on the general axioms. Their proofs are given under the head of *Inequalities* in nearly all Algebras, so they will not be considered here. They are used for those magnitudes for which equality axioms 2–5 are used.

(1) *If equals are added to, taken from, multiplied by, or divided into, unequals, the results are unequal in the same sense.*

(That is, the greater quantity remains greater after the operation is performed.)

(2) *If unequals are taken from, or divided into equals, the results are equal in the opposite sense.*

(3) *If unequals are added to, or multiplied by, unequals in the same sense, the results are unequal in the same sense.*

(4) *If the first of several magnitudes is greater than the second, the second greater than the third, the third greater than the fourth, and so on, then the first is greater than the last.*

NOTE. (4) holds for "the first less than the second," etc.; also any pairs might be equal without changing the result, if there is at least one inequality.

These statements are all in regard to positive magnitudes; they are not all true when negative quantities are used.

WARNING. *Unequals should not be taken from, or divided into, unequals, for the results cannot, in general, be determined.*

SECTION VII. PROPOSITIONS

84. Propositions. *Proposition* is a general term including:

(1) **Theorem,** which is a truth to be proved.

(2) **Corollary,** which is also a truth to be proved, but generally one that follows quite directly, and often very simply, from a known truth — most often from a theorem that has just been proved.

(3) **Problem, or Construction Theorem,** which requires that a certain figure be drawn from given parts. A more definite understanding of problem will be given in § 177.

85. Parts of a Theorem. A theorem is composed of the *condition*, or *hypothesis*, and the *conclusion*. It usually takes the form, — If a certain condition is true, a certain conclusion is also true. (See § 2.)

86. Proof of a Proposition. To prove any proposition, the student has certain materials from which to work, namely: the foregoing definitions, axioms, and truths concerning them; the theorems preceding the one that is being proved; and the condition of the proposition in question. From these known truths the proof must be deduced, and the required conclusion must be reached.

The proof must always be a logical one; the truth of a proposition must not be judged by measurements — as in Concrete Geometry — or from the appearance of the figure, for such methods have no place in this subject. It might be said, however, that a carefully drawn figure will sometimes give the *idea* which suggests the correct proof, although the appearance of the figure cannot be quoted as authority for the truth of any statement.

87. Order of Proof. The following order of proof has been found very convenient, and the student is advised to follow it in all work.

STATEMENT OF THEOREM

Figure

Given. *Condition*, in terms of the letters of the figure.

To prove. *Conclusion*, in terms of the letters of the figure.

Proof. I. The proof, in numbered steps, with the authority for each step following it.

NOTES. Any special case or interesting fact concerning the theorem.

COROLLARIES. Those connected with the theorem.

Examples of this form will be found in the proofs given later.

88. Classification of Theorems. All theorems of Plane Geometry may be divided into certain groups which might be called *classes;* for example, some theorems prove angles equal, others prove lines unequal, still others prove triangles congruent, etc.

It is evident that any theorem of a certain class must depend upon *something which will prove the particular result desired;* that is, *upon something in its own class, — unless it can be obtained by logic from another class.* Except in the cases of logic, which are readily recognized, each geometric truth depends directly upon *some preceding truth of its own class*, and the foundation truths of each class are the definitions and axioms that concern the things used.

89. Important Classes Already Started.

(1) **Congruence.** To prove figures congruent, they must be shown to coincide; the axiom of motion can be used to suppose one of two figures placed on the other in order to test the coincidence.

(2) **Angles Equal.** Right angles, straight angles, perigons, complements of equal angles, supplements of equal angles, explements of equal angles, vertical angles.

(3) **Magnitudes Unequal.** The inequality axiom: after having obtained one inequality, the inequality statements can be used. It follows from this that the only way to prove two things unequal at this stage of the work is to show that one is a part of the other, or that one equals a part of the other.

(4) **Line Straight.** A line is straight if its sects are the arms of a straight angle; the bisectors of vertical angles lie in a straight line.

(5) **Lines Perpendicular.** If the adjacent angles formed by them are equal; if they bisect adjacent supplemental angles.

(6) **Lines Coincide (or are determined).** If they go through the same two points, if they bisect the same angle, if they are perpendicular to the same line at the same point (in a plane).

(7) **Points Coincide (or are determined).** If they are intersections of the same two lines, if they are midpoints of the same sect, if they are correspondingly placed points of equal sects which are made to coincide.

(8) **Lines Equal.** This would be a case of congruence; the lines would have to be shown to coincide.

These are, of course, not all the classes to be found in Plane Geometry, but they serve to show that a foundation has already been laid, upon which certain classes of geometric truths can be established.

90. Method of Attack.

(1) Determine the class of the theorem.

(2) List the known ways of proving the required conclusion (or, in other words, list the ones of the same class).

(3) Examine each way found as to the possibility of its being applied to the *figure* in question, and especially to *conditions* of the theorem.

(4) Having decided on one or more ways as possibilities (probably by eliminating those which seem impossible of application), try to reason from the condition to the conclusion by the method chosen.

WARNING. *Remember that it is necessary to obtain the conclusion,* **and to use the condition**; *more trouble is made by neglect to think of these two points than by anything else in Geometry.*

91. ORAL AND REVIEW QUESTIONS

Define contraposite, straight line, adjacent angles, converse, vertical angles, axiom, perigon, right angle, complements, theorem. Are two supplemental angles ever equal? always equal? What is the complement of one half a right angle? of a right angle? of a zero angle? of three quarters of a right angle? What is the supplement of a right angle? of a right angle and a half? of two right angles? What does a perpendicular do to a straight angle? Why are all straight angles equal? all right angles? all perigons? How many points are needed to fix a line? How many lines are needed to fix a point? How many bisecting points can a sect have? Why? How many bisecting lines can an angle have? Why? How does the angle case apply to perpendiculars? What method of proof applies to proving that complements, supplements, and explements of equal angles are equal? Apply one of these to show that vertical angles are always equal. State the equality axioms; the straight-line axiom; the division axiom; the substitution axiom; the intersection axiom. Upon what foundation must the proofs of the propositions be based? If a statement is known to be true, what related statement is always true? What statement is sometimes true, and when? State the obverse of "This statement is true." Of what is a straight angle the sum? a perigon? Of what two equal angles is a straight angle the sum? a perigon? Of what four equal angles is a perigon the sum? If an angle is twice its complement, how large is the angle? If an angle is three times its explement, how large is it? Upon what does the size of an angle depend? What ways are known to prove figures congruent? angles equal? angles unequal? lines equal? lines unequal? lines perpendicular? lines not perpendicular? How can a line be determined? a point be determined? What order of proof can be used for a theorem? Define each of the parts of the proof. Distinguish between the terms "proposition," "theorem," "problem."

SECTION VIII. TRIANGLE THEOREMS

92. Theorem I. *If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, the triangles are congruent.*

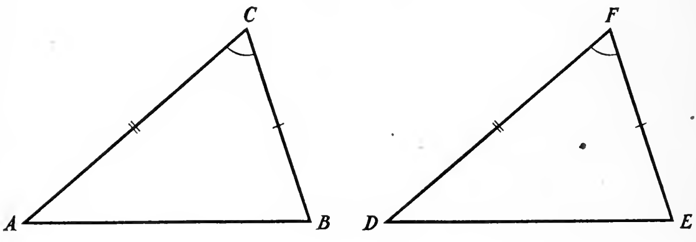
Analysis

Class. Triangles congruent.

Known Methods. Coincidence, using the axiom of motion.

Method to be used. One triangle will be supposed to be placed on the other, and the given facts will then be used to determine whether they would coincide.

Proof



Given. $\triangle ABC$, $\triangle DEF$; $BC = EF$, $CA = FD$, $\angle C = \angle F$.

To prove. $\triangle ABC \cong \triangle DEF$.

Proof. I. Suppose $\triangle ABC$ to be placed on $\triangle DEF$, C on F, and BC along EF (ax. of motion).

Then CA would lie along FD, $\because \angle C = \angle F$ (given).

II. B would fall on E, $\because BC = EF$ (given).
A would fall on D, $\because CA = FD$ (given).

III. AB would coincide with DE (but one straight line through two points).

IV. $\therefore \triangle ABC \cong \triangle DEF$ (def. $\cong$).

NOTE. It often makes the conditions of the theorem more clear if those conditions are indicated in the figure. The usual way of showing equal parts is to place a like mark on any two parts that are known to be equal. In the figure used in Th. I, the equal parts are indicated by such marks. Where equal parts are used in the theorem, although not given, the same method is sometimes used.

93. Corresponding Parts of Congruent Figures. When two figures coincide, each part (side or angle) of one coincides with a part of the other, and is therefore equal to it. Two parts of congruent figures that would coincide if the figures were made to coincide are called *corresponding*, or *homologous* parts.

When two figures are known to be congruent on account of their having certain equal parts,—as by Th. I,—the other corresponding parts can be told by their position relative to the known parts; as, by their being opposite to known parts, or between two known parts. In Th. I, $AB = DE$, $\angle A = \angle D$, $\angle B = \angle E$.

The most important use of congruence of figures is to prove equality of lines and of angles.

94. Theorem II. *If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, the triangles are congruent.*

ANALYSIS

This theorem is of the same class as Th. I, so must be done in the same way, or else by means of Th. I. It can be proved in either of these ways, and the superposition method is much the easier. It is so much like Th. I that the proof will be left for the student to do for himself.

EXERCISES

1. Prove Th. II by means of Th. I.

HINT. *A second side is all that is needed. Suppose that the second side of one is not equal to the corresponding side of the other, and cut off a sect on it which is equal to the corresponding side of the other, thus forming a triangle which is congruent to the second triangle. Now examine the angles in the figure.*

2. A point in the perpendicular bisector of a sect is equidistant from the ends of the sect.

3. In an equilateral triangle, the bisector of any angle forms two congruent triangles.

4. If the diagonals of a quadrilateral bisect each other, the quadrilateral has two pairs of equal sides.

5. If a diagonal of a quadrilateral bisects two angles, the quadrilateral has two pairs of equal sides.

WARNING. *In exercises 4 and 5, be certain that the sides used are corresponding parts of the triangles found.*

6. In a regular pentagon, the diagonals are equal.

7. If a line is drawn from the end of a line sect to a point in its perpendicular bisector, the line from the other end of the sect making the same angle with the sect will meet the perpendicular bisector at the same point.

8. If two triangles are congruent, medians drawn to corresponding sides are equal.

9. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

10. Two equal lines AC and AD are drawn on opposite sides of AB, making equal angles with AB. Prove that BC and BD will also make equal angles with AB.

11. If two triangles are congruent, the bisectors of corresponding angles are equal.

12. If one line is perpendicular to a second line, then two lines drawn from a point in the first line to the second line are equal if they make equal angles with the first line.

NOTE. It was said in § 93 that the most important use of congruent figures was to prove lines equal and angles equal. It is often necessary to choose between different congruent theorems, and the result desired affects greatly the choice of the theorem.

For example, if lines are to be proved equal, one is much more likely to be able to use Th. II than Th. I, for it requires less knowledge about equal lines — the thing that is being worked for. So, if angles are to be proved equal, Th. I is better than Th. II, for one is more likely to know about the lines than about the angles when the desired end is equality of angles.

Such considerations as the above are often of use in finding the proof of a theorem, and the pupil should give the question under examination very careful thought before attempting to write the proof.

95. Theorem III. *If a side of a triangle is extended, the exterior angle formed is greater than*

(1) *the angle opposite the side extended;*

(2) *either angle not adjacent.*

[Note that (2) includes (1).]

NOTE. This theorem is a hard one to discover at the beginning of Geometry. It is, however, a very useful theorem, and a very complete example of the classification method of finding a construction and proof. The pupil need not feel discouraged by its apparent difficulty, but should rather regard this analysis as an example of the method to be applied to the following theorems, most of which are much less difficult.

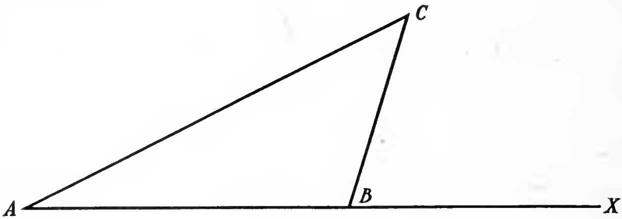
The proof of this theorem will be easy to understand after the auxiliary lines (or extra lines which must be added to the figure in order to obtain the proof) are found. Notice that the classification method shows what additional lines are needed in the figure, when the given figure does not itself include all the material necessary.

Analysis. (Part I)

Class. Angles unequal.

Preceding Methods. Inequality axiom.

Application.



It is necessary to show that $\angle XBC > \angle C$; that is, that $\angle C$ is part of $\angle XBC$, which is evidently untrue, or that $\angle C$ is *equal to part of* $\angle XBC$. Since this requires angles equal, a new class must be used.

Class. Angles equal.

Preceding Methods. Those in the list in § 89, (2); corresponding parts of congruent triangles.

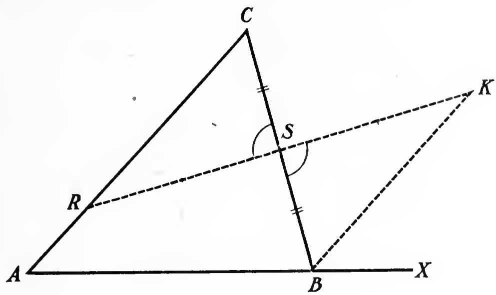
Elimination. $\angle C$ is not known to be a rt. $\angle$, st. $\angle$, or a perigon; there are no equal angles of which $\angle C$ and part of $\angle XBC$ could be complements, supplements, or explements; it is not vertical to any angle at B. There remains only the possibility that $\angle C$ and an angle at B may be corresponding angles of congruent triangles.

This requires still a third classification.

Class. Triangles congruent (triangles, because no applicable way of proving other figures congruent is known as yet).

Preceding Methods. 2 s. incl. $\angle$; 2 $\measuredangle$ incl. s. Probably 2 s. incl. $\angle$ will be used because angles are to be proved equal.

Application. Any line meeting BC and CA, as RS, will form a triangle of which $\angle C$ is a part, and any line from



B to the extension of RS, as BK, will form a triangle containing a part of $\angle XBC$. It remains only to draw these lines in such a way that the triangles formed will be congruent. Since $\angle C$ is to be corresponding to $\angle KBS$, the opposite sides must be made equal, and this can be done by cutting off $SK = RS$.

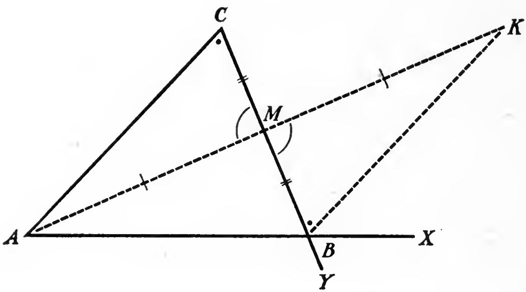
But $\angle CSR = \angle KSB$ (vert.), and the only other lines needed are CS and SB, for these, with the parts named, would make two sides and the included angle. As RS can be drawn anywhere, let S be the midpoint of BC.

Therefore, to make the triangles congruent, but two things are needed: That a line be drawn from any point on CA through the midpoint of BC, and extended its own length, the extremity being joined to B.

The proof has now been discovered *if $\angle KBS$ is a part of* $\angle XBC$. It is a part of it if it lies within $\angle XBC$; that is, if the point K is between BX and BC. K does lie between these arms unless line RSK meets one arm of $\angle XBC$ after intersecting BC at S. It cannot meet BC again (two st. lines meet in but one point), and it cannot meet BX again *if it has already met it.* The simplest way to have it meet

BX and still be drawn from a point on CA is to draw it from A. (See § 96.)

PROOF



Given. $\triangle ABC$; AB extended to X.

To prove. (1) $\angle XBC > \angle C$; (2) $\angle XBC > \angle A$.

Proof. I. Draw AK from A, through M, the midpoint of BC, to K, so that $MK = AM$. Draw BK.

II. $CM = MB$ (bisection).
$\angle CMA = \angle KMB$ (vert.).
$AM = MK$ (const.).
$\therefore \triangle CMA \cong \triangle KMB$ (2s. incl. $\angle$).

III. $\therefore \angle C = \angle KBM$ (cor. pts.).

IV. But $\angle KBM$ is part of $\angle XBC$ (AMK having met both arms).
$\therefore \angle XBC > \angle KBM$ (ineq. ax.).

V. $\therefore \angle XBC > \angle C$ (sub. ax.).

NOTE. This proves an exterior angle greater than the interior angle opposite the side produced.

VI. (2) Extend CB to Y; then $\angle ABY > \angle A$, since $\angle A$ is opposite the extended side CB (part 1).

VII. But $\angle CBX = \angle ABY$ (vert.).
$\therefore \angle CBX > \angle A$. (sub. ax.).

96. Location. That part of Th. III which shows that point K lies between the arms of $\angle XBC$, and all other parts of the Geometry where the place in which a point or line must lie is considered, are said to deal with the *location* of the thing in question. It is very important that the place of each point or line in the figure be fixed absolutely, as otherwise incorrect proofs will often be given. (See § 110.)

97. Cor. 1. *There can be but one perpendicular from a given point to a given line.*

Assume one perpendicular, and show that any other line from the same point will make an obtuse angle.

98. Determination of Lines. § 97 is another way to *determine*, or *fix* a line. Three ways have already been mentioned in § 89 (6); they are, by two points, by its bisecting an angle, by its being perpendicular to a line at a point in the line. The new one is by its being perpendicular to a line from a point outside the line. The last two are usually spoken of as perpendicular to a line *at* a point and *from* a point.

In making a construction line to help in obtaining a proof, the line can be drawn so as to do any one of these four things, *but no more than one.* For example, in Th. III, the line AM is drawn, and it is determined by A and M; then it is extended to K, so that $MK = AM$. This extension simply represents another part of the line determined by A and M, of which AM is one sect. The line KB is determined by the points K and B.

It should be noticed that the determination of a line usually depends upon the determination of one or more points; as in the case just discussed, the point A is given, and the point M is determined, because there is but one midpoint in a sect. *K* is determined because the sect *MK* equals the sect *AM*.

WARNING. *Never attempt to make a line do two determining things at the same time; as, bisect an angle and go from the vertex to a fixed point; or join two known points and be perpendicular to a fixed line. Always determine each line, but by one only of the determining conditions.*

The lines discussed in this paragraph — auxiliary lines — are only representations of lines that do exist. Each point or line added to a figure is the representation of some actual point or line that can exist in the figure, therefore it is necessary to be especially careful not to attempt to draw lines that are not possible.

99. COR. 2. *If a triangle has one right or obtuse angle, the other two angles are acute.*

Use Th. III.

13. In an isosceles triangle the base angles are acute.

14. The sum of any two angles of any triangle is less than a straight angle.

15. If from a point a perpendicular and other lines are drawn to a given line, then the angle formed by any of the lines on its side which is away from the perpendicular, is obtuse, and the greatest angles in the figure are the ones formed by the lines farthest from the perpendicular on its two sides.

16. If from the ends of a side of a triangle lines are drawn to a point within, the angle formed is greater than the angle opposite that side.

100. **Theorem IV**. *If two sides of a triangle are equal, the angles opposite those sides are equal.*

An auxiliary line is necessary, but the analysis will show what line is necessary. What method of proving angles equal uses the given, that is, uses equal lines?

17. An equilateral triangle is equiangular.

18. The bisectors of the base angles of an isosceles triangle are equal.

19. The medians to the legs of an isosceles triangle are equal.

20. Two points on the base of an isosceles triangle equally distant from the ends of the base are equally distant from the vertex.

21. If equal distances are laid off on the sides of an equilateral triangle taken in order, the points obtained are the vertices of an equilateral triangle.

22. Assuming that the bisectors of the base angles of an isosceles triangle meet, prove that they form an isosceles triangle.

101. Cor. 1. *In an isosceles triangle, the bisector of the vertex angle, the median to the base, the altitude to the base, and the perpendicular bisector of the base, are all one line.*

The proof of Th. IV can be extended to prove this. It is very largely a matter of determination of the lines.

102. Cor. 2. *There can be but two equal lines from a given point to a given line.*

Draw three lines supposed to be equal, and examine the base angles.

103. **Theorem V**. *If two sides of a triangle are unequal, the angles opposite those sides are unequal, the greater angle being opposite the greater side.*

The classification shows that it must be done by the inequality axiom or by exterior angle. Try to form a new angle equal to one of the angles or to part of it, but exterior to the other.

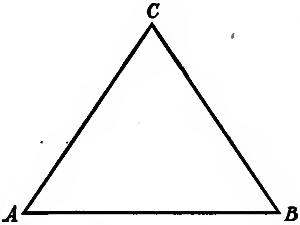
23. In the quadrilateral $ABCD$, if AB is greater than BC, BC than CD, and CD than DA, then angle D is greater than angle B. Is angle C always greater than angle A?

24. $ABCD$ is a quadrilateral of which DA is the longest side, BC the shortest. Prove angle B greater than D, and C than A.

104. Theorem VI. (*a*) *If two angles of a triangle are equal, the opposite sides are equal.* (*b*) *If two angles of a triangle are unequal, the opposite sides are unequal, the greater side being opposite the greater angle.*

This is evidently the converse of Th. IV and Th. V, and since they cover all possibilities, the law of converse can be used. The following form is recommended for all theorems proved by the law of converse.

PROOF



Given. $\triangle ABC$; $\angle A = \angle B$, $\angle A > \angle B$, or $\angle A < \angle B$.

To prove. $BC = CA$, $BC > CA$, or $BC < CA$, respectively.

Proof. I. If $BC = CA$, then $\angle A = \angle B$ (opp. = sides).
If $BC > CA$, then $\angle A > \angle B$ (opp. $\neq$ sides).
If $BC < CA$, then $\angle A < \angle B$ (opp. $\neq$ sides).

II. These statements cover all possibilities, and no two of the conclusions can be true at once.

III. $\therefore$ If $\angle A = \angle B$, then $BC = CA$.
If $\angle A > \angle B$, then $BC > CA$.
If $\angle A < \angle B$, then $BC < CA$ (law of converse).

25. An equiangular triangle is equilateral.

26. In a right-angled triangle, the hypotenuse is the longest side.

27. In an obtuse-angled triangle, the side opposite the obtuse angle is the longest side.

28. Assuming that the bisectors of the angles of a triangle which is not isosceles meet, prove the bisector of the smaller angle the longer line.

29. Angle A of triangle ABC is bisected by a line meeting BC at P. Prove that AB is longer than PB, CA than PC.

30. If a perpendicular and two other lines on the same side of the perpendicular are drawn from a point to a given line, the line which is farther from the perpendicular is the longer.

105. Theorem VII. *If two triangles have three sides of one equal respectively to the three sides of the other, the triangles are congruent.*

Will superposition work? If not, what other methods of proving triangles congruent are known? Don't neglect the *given*. Read Appendix, § 344 (2).

31. In an equilateral quadrilateral, (1) the diagonals bisect the angles and are the perpendicular bisectors of each other; (2) the opposite angles are equal.

32. Two triangles are congruent if they have two sides and the median to one of those sides respectively equal.

33. If one diagonal of a quadrilateral divides it into two isosceles triangles, the other diagonal bisects two of the angles and is the perpendicular bisector of the first diagonal; also, one pair of opposite angles are equal.

106. Theorem VIII. *If two right triangles have the hypotenuse and a side of the one respectively equal to the hypotenuse and a side of the other, the triangles are congruent.*

Note. Investigate also the congruence of triangles having two sides and an angle not included equal, when the angle is not right.

34. If two isosceles triangles have the equal sides of one the same length as the equal sides of the other, and the base of one double the altitude of the other, the triangles are equivalent.

35. If a quadrilateral has a pair of opposite sides equal, and a pair of opposite angles right angles, one diagonal divides it into congruent triangles.

107. Theorem IX. *The sum of any two sides of a triangle is greater than the third side.*

Draw the sum of the two sides in the simplest position possible; then classify.

108. Cor. I. *The difference of any two sides of a triangle is less than the third side.*

If $AB + BC > CA$, why is $CA - BC < AB$?

109. Cor. II. *A straight line between two points is less than any broken line between those points.* (See Appendix, § 346.)

36. If two sides of a triangle are 6 ft. and 10 ft. long, what can be told about the length of the third side?

37. If two sides of a triangle are a and b, what can be told of the length of the third side?

38. The sum of the sides of a triangle is greater than two thirds the sum of the medians, but is less than twice that sum.

39. The sum of the sides of a quadrilateral is greater than the sum of the diagonals, but is less than twice their sum.

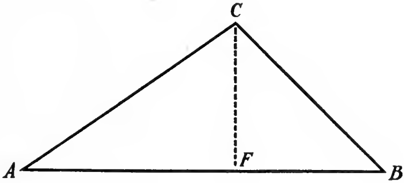
40. If from the ends of a side of a triangle lines are drawn to a point within, the sum of the lines so drawn is less than the sum of the other two sides of the triangle.

41. The sum of the lines from the vertices of a triangle to any point is greater than half the perimeter of the triangle.

42. The difference of the diagonals of a quadrilateral is less than the sum of either pair of opposite sides.

110. Incorrect Proofs depending on Location. — Many apparently correct proofs are entirely incorrect, or are incomplete, because some point or line *appears to be in a certain position, and is therefore assumed to be in that position in the proof*, when as a matter of fact it is never in that position at all, or is in that position for certain cases only.

If the assumed location is never right, the proof is entirely incorrect; if the location is right for some cases, but not for all, the proof applies only to those cases for which the location is correct, — that is, it is not a correct general proof. On this account great care should be taken that no point or line is assumed to be in a certain position unless it has been proved to lie in that position *always*.



EXAMPLE: INCOMPLETE PROOF OF THEOREM IX

Drop a perpendicular from C to AB at F.

Then $BC > FB$ (hyp. rt. $\triangle$).

$CA > AF$ (hyp. rt. $\triangle$).

And adding, $BC + CA > AF + FB$ (or AB).

The fault here is in assuming that $AF + FB = AB$, which is true only if F falls on AB. It is not necessary that F shall fall on AB, for the perpendicular can just as readily meet the extension of AB.

This proof can be completed so as to be correct for all cases by adding a proof that applies to the case where the point F is in the extension of AB.

111. Theorem X. *If two triangles have two sides of the one equal to two sides of the other, but the included angles unequal, then the third sides are unequal, the greater side being opposite the greater angle.*

Analysis

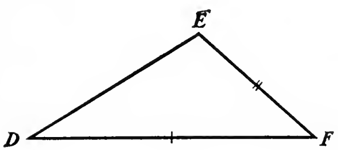
Class. Lines unequal.

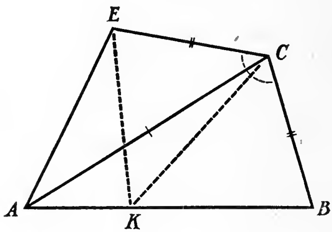
Preceding Methods. Ineq. ax.; opp. uneq. ∠s; 2 s. > 3 d.

Elimination. The inequality axiom has been used to obtain the two following ways, so it is more likely that it will be used indirectly through them than that it alone will give the proof. Neither of the other methods will apply if the triangles are entirely separate, so they must be put together, either by being placed side by side, or by superposition. The *given* will have to determine which method will be best. The fact that one angle is greater than the other can be used by cutting off the smaller on the greater, or by taking enough from the greater so that when it is added to the smaller, the angles will be equal. The second way is nothing more than bisecting the sum of the two angles. It is possible to get a proof by following any of the methods mentioned, but the following way is perhaps the easiest.

Given. $\triangle s\ ABC, DEF$; $BC = EF$, $CA = FD$, $\angle C > \angle F$.

To prove. $AB > DE$.





Application. Let the triangles be placed with FD coinciding with its equal CA, B and E lying on opposite sides of the common line. Draw a line bisecting the angle BCE; it will meet the opposite side AB of $\triangle ABC$, for

$\angle C > \angle F$, and so this line will lie within $\angle C$. Call the point where it meets AB, K.

Now the condition has changed from two sides equal and the included angles unequal, to two sides (BC and CK, and EC and CK) and the included angles ($\angle BCK$ and $\angle ECK$) *equal*. This suggests congruent triangles, so the line EK is drawn, forming $\triangle ECK \cong \triangle BCK$. Then $EK = KB$, and the proof by two sides greater than the third follows.

NOTE. This proof holds even when EA and AB happen to fall in one straight line.

WARNING. "opp. $>$ $\angle$," and "opp. $>$ s." *are often used as authority. It must be kept in mind that this applies only* **in the same triangle, or in triangles having two sides equal.** *Lines and angles must not be judged unequal on account of the opposite parts in any other cases.*

43. In triangle ABC, if CA is greater than AB, and P and Q are taken on AB and CA, respectively, so that PB equals QC, prove BQ less than CP.

44. If a quadrilateral has a pair of opposite sides equal, but the angles formed by those sides with one of the other sides unequal, then the diagonals are unequal.

45. If no median of a triangle is perpendicular to a side, then the triangle has no equal sides.

46. If an equilateral pentagon is not equiangular, the longest diagonal is that which joins the vertices on either side of the greatest angle.

112. Theorem XI. *If two triangles have two sides of the one equal to two sides of the other, but the third sides unequal, then the angles opposite those sides are unequal, the greater angle being opposite the greater side.*

What relation has this to Th. X? Is Th. X all that is required to prove this theorem?

47. In the figure of exercise *44*, the angles formed by the equal sides with the fourth side are also unequal, the greater angle being opposite the greater of the two given angles.

48. If lines are drawn from the ends of the base of a triangle so as to cut off equal distances (from the base) on the opposite sides, the triangle is isosceles when those lines are equal; when those lines are unequal, the longer line is drawn to the shorter side.

113. Theorem XII. *Of all lines drawn to a given line from a given external point*

(*a*) *the perpendicular is the shortest.* (See Appendix, § 346.)

(*b*) *those making equal angles with the perpendicular, or cutting off equal distances from its foot, are equal.*

(*c*) *those making unequal angles with the perpendicular, or cutting off unequal distances from its foot, are unequal, the one making the greater angle, or cutting off the greater distance, being the greater.*

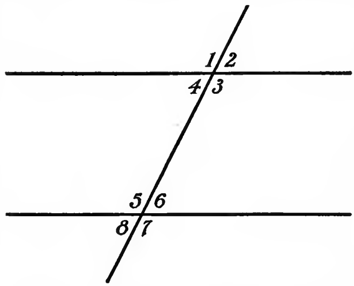
114. Cor. 1. (*a*) *Equal obliques from a point to a line make equal angles with the perpendicular and cut off equal distances from its foot.*

(*b*) *Unequal obliques from a point to a line make unequal angles with the perpendicular, and cut off unequal distances from its foot, the longer oblique making the greater angle, and cutting off the greater distance.*

SECTION IX. PARALLELS AND PARALLELOGRAMS

115. Angles formed by Two Lines and a Transversal. A line cutting other lines is called a *transversal* of those lines.

If two lines are cut by a transversal at two points, eight angles are formed at those points, and certain sets of those angles have names as follows:



The two sets of angles in the figure — $\measuredangle$s 1, 3, 5, 7 and $\measuredangle$s 2, 4, 6, 8 — are called *transverse sets* of angles, and any two angles in the same set are called *transverse angles.*

The angles $\angle 1$, $\angle 2$, $\angle 7$, $\angle 8$ are called *exterior* angles, and the angles $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$ are called *interior* angles.

The pairs of angles $\angle 1$ and $\angle 7$; $\angle 2$ and $\angle 8$; $\angle 3$ and $\angle 5$; $\angle 4$ and $\angle 6$ are called *alternate pairs of angles.*

The pairs $\angle 1$ and $\angle 5$; $\angle 2$ and $\angle 6$; $\angle 3$ and $\angle 7$; $\angle 4$ and $\angle 8$ are called *corresponding* or *exterior interior* angles. It is evident that any pair of alternate or corresponding angles are also in the same transverse set.

116. Theorem XIII. *If a transversal of two lines makes one pair of transverse angles, which are not vertical, equal,*

(1) *any two transverse angles are equal;*

(2) *any two angles not transverse are supplemental.*

117. Cor. 1. *If two non-adjacent angles, which are not transverse, are supplemental,*

(1) *any two transverse angles are equal;*

(2) *any two angles not transverse are supplemental.*

Note. These propositions are introductory to the subject of parallels, and should be used wherever possible to shorten the following proofs. They show that the angles formed by two lines and a transversal separate into two groups when there is any question of equality of the angles, and that the angles of different groups are supplemental when angles of the same group are equal.

118. Parallels. Two straight lines in the same plane which never meet, however far produced, are called *parallel lines.*

119. Parallel Axiom. *Through a given point there can be but one parallel to a given line.*

This is sometimes stated: *Two intersecting lines cannot both be parallel to the same line.*

***120.** *Lines parallel to the same line are parallel to each other.* For, if two lines that are parallel to the same line should meet, there would be more than one parallel through a point.

121. Theorem XIV. *If a transversal of two lines makes*

(1) *a pair of transverse angles, which are not vertical, equal, or*

(2) *two non-adjacent angles, which are not transverse, supplemental,*

then the two lines are parallel.

Is there any difference between the two cases? If the lines meet, examine the angles; use contraposite.

122. Cor. 1. *If a transversal of two lines makes*

(1) *a pair of transverse angles unequal,*

(2) *two angles which are not transverse not supplemental, then the lines are not parallel.*

This is the obverse of § 121. The best way to prove the obverse of a single statement is to use both "givens," and show that the two conclusions could not be the same. Applied to this statement, the work would be about as follows :

If a pair of transverse angles were equal, the lines would be parallel.

If a second line through the same point on the transversal makes an angle not equal to the other of the pair, that line cannot also be parallel, by the parallel axiom. This is, of course, in very condensed form ; the proof should be in terms of the letters of the figure.

Note that the lines, if not parallel, meet on the side of the smaller of the two alternate interior angles. Why ?

123. Cor. 2. *Lines perpendicular to the same line are parallel.*

49. The bisector of the exterior vertex angle of an isosceles triangle is parallel to the base.

50. If AC and BD bisect each other, prove that AB is parallel to CD.

51. The bisectors of two consecutive angles of any polygon intersect one another.

52. If all the angles of a quadrilateral are right angles, the figure has two pairs of parallel sides.

124. Theorem XV. *If two parallels are cut by a transversal,*

(1) *any two transverse angles are equal;*

(2) *any two angles that are not transverse are supplemental.*

What relation has this to Th. XIV? to Th. XIV, Cor. 1? After proving this by the simplest way, see Appendix, § 341.

125. Cor. 1. *A line perpendicular to one of two parallels is perpendicular to the other.*

It is necessary first to prove that it meets the other; in other words to prove that it cannot be parallel to it. If it were parallel to one of the two, while given perpendicular to the other, what impossible figure would be formed?

Warning. *Never assume that two lines meet without proving why they meet.*

126. Cor. 2. *Two lines that are perpendicular to two intersecting lines are not parallel.*

127. Cor. 3. *If the arms of one angle are parallel to the arms of a second angle, the angles are equal, or supplemental (according to their relative positions).*

128. Cor. 4. *If the arms of one angle are perpendicular to the arms of a second angle, the angles are equal, or supplemental (according to their relative positions).*

Through the vertex of one of the angles draw parallels to the arms of the other angle, thus using angles at one vertex (§ 127).

53. The bisectors of a pair of alternate angles formed by parallels with a transversal are parallel.

54. A line through the vertex of an isosceles triangle parallel to the base bisects the exterior angle at the vertex.

55. If the bisector of an exterior angle of a triangle is parallel to a side of the triangle, the triangle is isosceles.

56. If from a point in the base of an isosceles triangle, lines are drawn parallel to the equal sides, those lines form, with the equal sides, a quadrilateral whose perimeter equals twice one of the equal sides.

57. A line parallel to the base of an isosceles triangle makes equal angles with the legs, or the legs produced.

58. A line parallel to the base of an isosceles triangle is perpendicular to the bisector of the vertex angle.

129. Theorem XVI. *In any triangle,*

(1) *any exterior angle equals the sum of the two non-adjacent interior angles;*

(2) *the sum of all the interior angles equals a straight angle.*

What is known about the exterior angle? Use it.

130. Cor. 1. *If two triangles have a side and any two angles of the one equal to a side and the respective angles of the other, the triangles are congruent.* Use the theorem.

This is the last of the propositions on the congruence of triangles by equal parts. It should be noticed that three parts are always necessary, and that any three corresponding parts will prove the triangles congruent except

(1) three angles. (See §§ 286, 289.)

(2) two sides and an acute angle not included (called the *Ambiguous Case*). (See § 106.)

131. Cor. 2. *If a triangle has one right angle, the other two angles are complements; in an equilateral triangle, each angle is one third of a straight angle.*

59. Using (1) of § 129, prove (2) by drawing a line from the vertex to any point in the base.

60. The bisectors of two interior angles of a triangle meet (examine the angles made with the included side), and form an angle equal to a right angle plus one half the third angle.

61. The bisectors of two exterior angles of a triangle meet, and form an angle equal to one half the sum of the interior angles supplemental to the ones bisected.

62. If an angle of a triangle is bisected, and a line is drawn perpendicular to that bisector, that line makes an angle with either arm of the bisected angle equal to half the sum of the other angles of the triangle, and makes an angle with the third side equal to half the difference of those angles.

63. Any two altitudes of a triangle make equal angles each with the side to which the other is drawn.

64. Find the sum of the angles of a quadrilateral by drawing a diagonal.

132. Theorem XVII. *The sum of the interior angles of a polygon of n sides is $(n-2)$ straight angles.*

65. What is the sum of the interior angles of a pentagon? a hexagon? a decagon? an octagon? a 29-sided figure?

66. How many sides has the polygon the sum of whose interior angles is 12 st. angles? 3 st. angles? 17 st. angles?

67. How large is one angle of a regular pentagon? hexagon? decagon? octagon? 34-sided figure?

133. Theorem XVIII. *The sum of the exterior angles of a polygon is two straight angles.*

68. How many sides has the polygon the sum of whose interior angles equals the sum of the exterior angles?

69. How many sides has a polygon if the sum of the interior angles is seventeen times the sum of the exterior angles?

70. How many sides has the polygon one third the sum of whose interior angles is ten times the sum of its exterior angles?

71. How many sides has the polygon each of whose exterior angles is one tenth of a perigon?

72. How many sides has the polygon each of whose interior angles is six sevenths of a straight angle?

134. Degrees, Minutes, and Seconds. For convenience in numerical calculations, a perigon is divided into 360 equal parts, called *degrees;* each degree is divided into

60 equal parts called *minutes*, and each minute into 60 equal parts called *seconds.*

73. How many degrees are there in an interior angle of a regular polygon of 15 sides? of n-sides?

74. How many degrees are there in an exterior angle of a regular polygon of 20 sides? of n-sides?

135. Quadrilaterals having Two or More Parallel Sides. A quadrilateral having two pairs of parallel sides is called a *parallelogram;* one which has one pair of parallel sides is called a *trapezoid;* one which has no parallel sides is called a *trapezium.*

The parallel sides of a trapezoid are called its *bases*, the other two sides its *legs.*

Either pair of parallel sides may be called the *bases* of a parallelogram, but the side on which it appears to stand is usually called its *base.*

A parallelogram having all its angles right angles is called a *rectangle;* one having all its sides equal is called a *rhombus;* a rectangular rhombus is called a *square.* A parallelogram that is neither a rectangle nor a rhombus is called a *rhomboid.*

136. Theorem XIX. *Any two consecutive angles of a parallelogram are supplemental, and any two opposite angles are equal.*

137. Cor. 1. *A parallelogram having a right angle is a rectangle.*

138. Theorem XX. *In any parallelogram,*

(1) *either diagonal divides it into congruent triangles;*

(2) *the opposite sides are equal.*

139. Cor. 1. *The diagonals of a parallelogram bisect each other.*

75. The diagonals of a parallelogram are equal if the figure is a rectangle.

76. A parallelogram having two consecutive sides equal is a rhombus.

77. Two trapezoids are congruent if their sides are respectively equal.

140. Theorem XXI. *A quadrilateral is a parallelogram if*

(1) *two opposite sides are equal and parallel; or*

(2) *the opposite sides are equal.*

What is the definition of a parallelogram? What ways of proving this do you know? Keep in mind that the *given* must be used.

141. Theorem XXII. *A quadrilateral is a parallelogram if the opposite angles are equal.*

142. Theorem XXIII. *A quadrilateral is a parallelogram if the diagonals bisect each other.*

78. Is a quadrilateral a parallelogram if one diagonal divides it into congruent triangles? if each diagonal divides it into congruent triangles?

79. If each side of a square is extended from each end a length half as long as the diagonal, the figure formed by joining the ends of the sects obtained is a regular octagon.

80. On diagonal BD of parallelogram $ABCD$, points K and L are taken so that $BK = LD$. Prove that $AKCL$ is a parallelogram.

81. Perpendiculars from opposite vertices of a parallelogram to the diagonal joining the other vertices are equal.

82. Lines through the vertices of a triangle parallel to the opposite sides form a triangle having sides twice the length of the sides of the given triangle.

83. Lines drawn through the vertices of a quadrilateral parallel to the diagonals form a parallelogram twice as large as the quadrilateral.

84. Any quadrilateral is equal to a triangle having as sides the diagonals of the quadrilateral, and the included angle equal to the angle between the diagonals.

85. The sum (or the difference) of the perpendiculars from one pair of opposite vertices of a parallelogram to any line equals the sum (or the difference) of the perpendiculars from the other vertices to the same line.

86. The sum of the perpendiculars dropped from a point in the base of an isosceles triangle to the legs is constant.

87. The difference of the perpendiculars from any point in the extension of the base of an isosceles triangle to the legs is constant.

143. Theorem XXIV. *If two or more sects cut off on one transversal by parallels are equal, the corresponding sects cut off on any other transversal by the same parallels are also equal;* including as a special case,

A line through the midpoint of a side of a triangle, parallel to a second side, bisects the third side.

144. Cor. 1. *A line joining the midpoints of two sides of a triangle is parallel to the third side, and equals one half that side.*

145. Cor. 2. *If through the vertices of a triangle, lines are drawn parallel to the opposite sides, a new triangle is formed, such that the sides of the given triangle join the midpoints of the sides of the new triangle.*

88. The lines from two opposite vertices of a parallelogram to the midpoints of two parallel sides trisect the diagonal joining the other two vertices.

89. The lines joining the midpoints of the opposite sides of a quadrilateral bisect each other.

When do lines bisect each other? What way is there of using *midpoints* to obtain this?

90. The lines of *89* intersect at the midpoint of the line which joins the midpoints of the diagonals.

91. A line through the midpoint of one of the legs of a trapezoid, parallel to the bases, bisects the other leg.

92. The line joining the midpoints of the legs of a trapezoid is parallel to the bases.

93. What line passes through the midpoints of all the lines drawn from a point to a line?

94. If, in an isosceles triangle, any number of parallels to the base are drawn, the perpendicular bisector of the base goes through the intersections of the diagonals of each trapezoid formed.

NOTE. The trapezoid, while not used in the theorems, has been taken up quite frequently in exercises. Many of the triangle theorems have corresponding facts for trapezoids, and on this account, a good way to study the trapezoid is to word the triangle theorems so that they apply to trapezoids, and investigate their truth. Probably the most valuable auxiliary line for trapezoids is the line from the vertex between the shorter base and a leg, parallel to the other leg. This line divides the trapezoid into a triangle and a parallelogram, and so allows the theorems for those figures to be applied.

SECTION X. LOCI OF POINTS AND CONCURRENCE

146. Locus. That place, or places, in which *all points which fulfill a certain condition and no others* lie, is called the **locus** of points which fulfill that condition.

If all pupils of a certain class are in a certain room (that is, no pupil is absent), and all in the room belong to that class (that is, there are no visitors present), then that room contains *all pupils of that class and no others.* Similarly, if a line contains every point of a certain kind (where "kind" means in respect to some condition), and every point on it is of that kind, then it is the locus of such points. Evidently a locus is nothing but the place which may be said to belong to a certain kind of point.

To prove a locus, it is necessary to prove two things:

(1) all points of the kind lie in the place;

(2) all points in the place are of the same kind.

These statements are converses of each other, and their contraposites will also be true.

(3) all points not in this place, are not of this kind;

(4) all points not of this kind do not lie in this place.

It is sometimes easier to prove (3) or (4) instead of its contraposite, the most common way being to prove (3) instead of (1).

147. Attack of a Locus Theorem. The best way to discover a locus theorem is to take a point of the required kind and study it for characteristics which will show that it must lie in some certain place. After the place is found, it is also necessary to prove that every point in the place is of the same kind.

148. Theorem XXV. *Find the locus of points equidistant from two given points.*

Take any point equidistant from two given points, join the three points, and what kind of a triangle will be formed? What will pass through the vertex? Then what is the *place* which appears to be the locus? Now prove that all points on the line found are equidistant from the given points.

149. Cor. 1. *A line containing two points that are equidistant from the ends of a sect is the perpendicular bisector of that sect.*

150. Cor. 2. *The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.*

95. Find the locus of points equidistant from three points. (Use the points two at a time.)

151. Concurrence. If two or more lines meet in a point, the lines are said to be *concurrent* at that point.

152. Theorem XXVI. *The perpendicular bisectors of the three sides of a triangle are concurrent.*

Show that two meet, then that the third must pass through the same point.

This point of concurrence is equidistant from the three vertices, and is called the *circumcenter* of the triangle. (See § 165.)

96. If the circumcenters of the triangles formed by a diagonal of a parallelogram are joined to the ends of the diagonal, a parallelogram is formed.

97. The circumcenters of the four triangles into which a quadrilateral is divided by its diagonals are the vertices of a parallelogram.

98. In triangle ABC, if the midpoints of BC and CA are joined, the circumcenters of the original triangle (S) and of the new triangle (S') and C are in one straight line, and S' is the midpoint of the line.

153. Theorem XXVII. *Find the locus of points equidistant from two intersecting lines.*

NOTE. Locus does not always mean equidistance, although it involves equidistance in the first few cases.

99. Find the locus of points equidistant from two parallel lines.

100. Find the locus of points at a fixed distance from a given line.

101. If two points between the arms of an angle are equidistant from those arms, the line through the points is the bisector of the angle.

102. Find the locus of points equidistant from three lines that intersect in no point; in one point; in two points; in three points.

103. Find the locus of points equidistant from two intersecting lines, and at the same time equidistant from two points.

104. Find the locus of points equidistant from two given lines and at a fixed distance from a given line.

154. Theorem XXVIII. *The bisectors of the interior angles of a triangle are concurrent.*

The point of concurrence is called the *incenter*, and it is equidistant from the three sides of the triangle. (See § 166.)

155. COR. 1. *The bisectors of any two exterior angles of a triangle, and the third interior angle, are concurrent.*

There are three such points of concurrence, all equidistant from the sides of the triangle; they are called *excenters.* (See § 166.)

Of what are the incenter and the excenters the locus?

105. If the incenter of an equilateral triangle is joined to the vertices, the triangles formed are congruent.

106. If the incenters of the triangles formed in *105* are joined to each other, the incenter of the original triangle is the circumcenter of the new triangle.

107. If the incenters of the triangles into which a diagonal divides a parallelogram are joined to the ends of the diagonal, a parallelogram is formed.

108. The incenters and the circumcenters of the triangles into which a diagonal divides a parallelogram are the vertices of a parallelogram.

156. Theorem XXIX. *The altitudes of a triangle are concurrent.*

Use § 145. The point is called the *orthocenter.*

109. In triangle ABC, if YX joins the midpoints of BC and CA, and O and O' are the orthocenters of triangles ABC and XCY, then O, O', and C are collinear, and $CO' = OO'$.

110. In the figure of *109*, if S and S' are the circumcenters, I and I' the incenters, then $O'S'$ is parallel to OS and equals one half OS, $O'I'$ is parallel to OI and equals one half OI.

157. Theorem XXX. *The medians of a triangle are concurrent in a trisection point of each.*

Show that two meet. Draw the line from the third vertex through their intersection, and show that it bisects the third side, and is therefore the third median. This line may be extended its own length past the intersection point, and shown to be one diagonal of a parallelogram.

The point of concurrence is called the *centroid.*

111. The sum of the three medians of a triangle is greater than three fourths the sum of the sides.

112. In triangle ABC, if YX joins the midpoints of BC and CA, and if T and T' are the centroids of triangles ABC and XCY, T, T', and C are collinear, and $CT' = TT'$.

113. In the figure of *112*, if S', I', O', and S, I, O, are the circumcenter, incenter, and orthocenter, respectively, of triangles XCY and ABC, then $T'S'$ is parallel to TS and equals one half of it; $T'I'$ is parallel to TI and equals one half of it; $T'O'$ is parallel to TO and equals one half of it.

158. SUMMARY OF THEOREMS AND COROLLARIES, BOOK I

(Numbers in parentheses refer to black-faced section numbers.)

I. Congruence of Triangles. Triangles are congruent when they have three parts (of which at least one is a side) of one equal to the corresponding parts of the other, *except* when the parts are two sides and an acute angle not included between those sides.

Cases. 2 s. incl. ∠ (92); 2 ∠s incl. s. (94); 3 s. (105); rt. ∠, hyp., leg (106); 2 ∠s any s. (130); diag. par. (138).

II. Lines are Determined, or Coincide. One ⊥ from a point (97); lines in an isosceles △ (101); one ∥ through a point (119). (See also 89.)

III. Lines Equal. Congruence; opp. = ∠s in a △ (104); obliques (113, 114); opp. s. ▱ (138); diag. ▱ bisect (139); ∥s cutting off equal sects on transversals (143); lines through midpt. of s. of a △ (144, 145); midpt. hyp. rt. △ equidist. (150).

IV. Lines Unequal. Opp. ≠ ∠s in a △ (104); but two = lines from a pt. (102); sum of two s. of a △ > 3d, diff. less (107, 108, 109); opp. ≠ ∠s in △s having two = s. (111); obliques (113, 114).

V. Angles Equal, and Angle Equations. Opp. = s. (100); obliques (114); ∥s (116, 117, 124); ∥ arms (127); ⊥ arms (128); opp. ∠s ▱ (136, 137). (See also 89.)

Ext. ∠ of a △ = two int. ∠s, sum ∠s of a △ = st. ∠ (129); acute ∠s of a rt. △ = rt. ∠ (131); int. ∠s of a polygon = $(n-2)$ st. ∠s (132); ext. ∠s = 2 st. ∠s (133).

VI. Angles Unequal. Ext. ∠ of a △ (95); in rt. or obtuse angled △ other ∠s acute (99); opp. ≠ s. in a △ (103); opp. ≠ s. in two △s having two = s. (112); obliques (114).

VII. Angles Supplemental. ∥s (116, 117, 124); ∥ arms (127); ⊥ arms (128); con. sec. ∠s of a ▱ (136).

VIII. Lines Parallel. $\measuredangle s =$ (121); $\measuredangle s$ sup. (121); $\perp$ same line (123); $\parallel$ same line (120); opp. s. $\square$. (See XI of this set); line through midpts. of the s. of a $\triangle$ (144.)

IX. Lines not Parallel. But one $\parallel$ through a pt. (119); $\measuredangle s \neq$ (122); $\measuredangle s$ not sup. (122); $\perp$ intersecting lines (126).

X. Lines Perpendicular. Line $\perp$ one of two $\parallel$s $\perp$ other (125); contains two points equidist. (149). (See also 89.)

XI. A Quadrilateral a Parallelogram. Opp. s. $\parallel$ (135); opp. s. $=$ (140); two opp. s. $=$ and $\parallel$ (140); opp. $\measuredangle s =$ (141); diag. bisect (142).

XII. Loci. Points equidist. from two points (148); equidist. from two intersecting lines (153). (See exercises *99*, *100*.)

XIII. Concurrence. $\perp$ bisectors of sides (152); bisectors int. $\measuredangle s$ (154); bisectors 2 ext. $\measuredangle s$ and int. (155); altitudes (156); medians (157).

These classifications should be carefully studied, and made use of in solving exercises depending on the respective classes. The pupil should not depend on the list here, but should know the important ways of doing theorems of each class.

159. ORAL AND REVIEW QUESTIONS

What are the two most common ways of proving lines equal? the three most common of proving angles equal? the ways to prove lines parallel? not parallel? a figure a parallelogram? Name the points of concurrence of the sets of lines in a triangle. What is known about the circumcenter? the incenter? the excenters? the centroid? If you wish to prove that two lines bisect each other, how ought you to start? If it is necessary to prove two lines equal, and two angles are given equal, how are they likely to be used? How can corresponding parts of congruent triangles be distinguished without superimposing? What would be the sum of the interior angles of a polygon of K sides? of 79 sides? the exterior angles of each? If the interior angles of a polygon have a sum of 29 straight angles, how many sides has the polygon? a sum of $(2K + 6)$ right angles? If the interior angles of a polygon are eleven times the exterior angles, how many sides has the polygon? If each exterior angle is one twenty-second of a straight angle, how many sides has the polygon? If each interior angle is nineteen twentieths of a straight angle, how many sides has the polygon? If two sides of a triangle are 17 and 28, how long is the third side? How many points need to be known to determine a perpendicular bisector? What ways are known to determine a line? In how many ways should the same line be determined? What class of theorems would be used to prove that two lines meet? Outline in as brief a manner as possible the methods used in proving the following: base angles of an isosceles triangle are equal; sum of interior angles of a polygon; quadrilateral having two pairs of equal sides is a parallelogram; lines perpendicular to the same line are parallel; but one perpendicular from a point to a line; exterior angle of a triangle equals the sum of the non-adjacent interior angles; opposite angles of a parallelogram are equal; opposite sides of a parallelogram are equal; angles having parallel arms are equal or supplemental; perpendicular bisectors of the sides of a triangle are concurrent.

NOTE. Be ready to explain *why* each construction used in these proofs is the best one to use.

If the sides of a triangle are a, b, c, give all the length relations possible between the sides. How many equal lines are there from a point to a line? Which statement (converse, obverse, or contraposite) always follows from a true statement? which sometimes follows? when? State three places where converse is used to prove theorems or corollaries. Try to explain why these propositions are not in reverse order. Give some examples of contraposite reasoning used in proving theorems. What is the locus of points equidistant from three points not in the same line? Can the locus of points equidistant from four or more points be found? Why? From four or more lines? Why? What are the two most important things to keep in mind in trying to prove a theorem? which should be thought of first, in general? Why is it necessary to be careful of the location of points and lines?

GENERAL EXERCISES

114. If two lines drawn from the ends of the base of an isosceles triangle to the opposite sides cut off equal distances from the vertex, those lines are equal.

115. If one angle of a triangle is twice a second angle, the bisector of the larger angle forms a triangle equiangular to the given triangle.

116. If one of the opposite sides of a quadrilateral is the longest side, the other the shortest side, examine the size of the opposite angles.

117. The sum of the lines from the vertices of a triangle to any point is greater than one half the perimeter of the triangle.

118. The sum of the lines from the vertices of a quadrilateral to any point is greater than the sum of the diagonals.

119. The difference of the diagonals of a quadrilateral is less than half the sum of its sides.

120. If two points K and L are on the same side of the line AB, to what point P in AB must lines be drawn from K and L, so that $KP + LP$ shall be the shortest possible length? (Use symmetry.)

121. If a pair of base angles of a trapezoid are equal, the legs are equal, and conversely.

122. If the legs of a trapezoid are unequal, the base angles are unequal, and conversely.

123. The diagonals of an isosceles trapezoid are equal.

124. The lines joining the midpoints of the consecutive sides of a quadrilateral form a parallelogram. Examine the special cases, such as parallelogram, rhombus, rectangle, square.

125. Find out everything possible about the diagonals of a rhombus.

126. The perpendicular bisector of a side of a triangle intersects the greater of the other two sides, if the triangle is not isosceles.

127. In a right triangle, the altitude to the hypotenuse divides the triangle into two triangles whose angles equal the angles of the original triangle.

128. The line through the midpoints of the bases of an isosceles trapezoid passes through the intersection of the diagonals, and the intersection of the legs.

129. Two lines through a point equidistant from two parallels cut off on the parallels, if they meet them, equal sects.

130. The midpoint of any transversal of two parallels is equidistant from them.

131. Lines drawn from any point in the bisector of an angle parallel to its arms form a rhombus with those arms.

132. A line through the incenter of a triangle parallel to a side equals the sum of the sects it cuts off on the other two sides.

133. If the three sides of one triangle are parallel respectively to the three sides of another triangle, the angles of the first triangle are respectively equal to the angles of the second triangle.

134. If the sides of one triangle are respectively perpendicular to the sides of a second triangle, the angles of the first triangle are respectively equal to the angles of the second triangle.

135. The angle between the bisectors of two exterior angles of a triangle equals the complement of half the third interior angle.

136. The angle between the bisectors of one interior and one exterior angle of a triangle equals half the interior angle at the third vertex.

137. If the vertex of an isosceles triangle is joined to the trisection points of the base, the middle of the three angles formed is the largest.

138. If one acute angle of a triangle is double another, the triangle can be divided into two isosceles triangles by a line through the third vertex.

139. Two exterior angles of a triangle less the third interior angle equals a straight angle.

140. The angle between the bisectors of two angles of a triangle equals the third angle plus half the sum of the bisected angles.

141. Two equilateral triangles are congruent if an altitude of one equals an altitude of the other.

142. An angle of a triangle is acute, right, or obtuse, according as the median from it is greater than, equal to, or less than one half the opposite side.

143. The angle between the bisectors of the base angles of an isosceles triangle equals the exterior angle at the base.

144. If equal distances are measured off on the sides of a square taken in order, the points obtained will be the vertices of a second square.

145. A transversal of the legs of an isosceles triangle, one produced, which is perpendicular to the base, forms an isosceles triangle with the legs.

146. Divide the right angle of a right triangle into parts equal to the acute angles, and so prove that the midpoint of the hypotenuse is equidistant from the three vertices.

147. If one of the equal sides of an isosceles triangle is extended through the vertex its own length, the line joining the end of the extension to the other end of the base is perpendicular to the base.

148. If the angles of a quadrilateral are bisected, the angles formed by the bisectors of any pair of adjacent angles is supplemental to the angle formed by the bisectors of the other angles.

149. A trapezoid whose diagonals make equal angles with a base is isosceles.

150. Prove the theorem for the sum of the angles of a polygon by joining one vertex to all the others. How many diagonals can be drawn in any n-sided polygon? How many triangles will be formed in an n-sided polygon?

151. How many sides has a polygon the sum of whose interior angles is four times the sum of the interior angles of a quadrilateral?

152. How many sides has a polygon the sum of whose interior angles is equal to the sum of the interior angles of a hexagon plus the sum of the exterior angles of an octagon?

153. The triangles formed by producing the sides of an equiangular hexagon through both vertices of each side are equilateral.

154. The sum of the angles formed by producing the sides of a hexagon both ways until they meet equals a perigon.

155. If the sides of any polygon of n-sides are produced from both ends until they meet, the sum of the angles so formed is $n-4$ straight angles.

156. If each consecutive pair of angles of a quadrilateral are supplemental, is the figure a parallelogram?

157. A parallelogram which has supplemental opposite angles is a rectangle.

158. Is a quadrilateral a parallelogram if one pair of opposite sides are equal, and the other pair are parallel?

159. If each side of a square is produced in order, the extensions of the opposite sides being equal, the points obtained are the vertices of a parallelogram.

160. The sum of the perpendiculars from any point to the three sides of an equilateral triangle is constant. (In certain cases some of the lines become negative, and the difference would be taken.)

161. If X and Y cut off equal sects from A and C on the sides AB and CD of the parallelogram $ABCD$, $AXCY$ is a parallelogram.

162. Parallelograms having two consecutive sides and an angle of one equal to the corresponding parts of the other are congruent.

163. Rectangles having two consecutive sides of the one equal to the corresponding parts of the other are congruent.

164. In any parallelogram, a line through the midpoint of a side parallel to a consecutive side divides it into two congruent parts.

165. The median to any side of a triangle is less than one half the sum of the other two sides.

166. The midpoints of two opposite sides of any quadrilateral and of its diagonals are the vertices of a parallelogram. What if the given figure is a parallelogram?

167. A line drawn from A of triangle ABC, through the midpoint of the median to CA, to K on BC, cuts off one third of BC.

168. If three parallels are drawn from the ends and the midpoint of a sect to meet a given line, the parallel from the midpoint is half the sum of the parallels from the ends.

169. If equal sects are cut off on one leg of an isosceles triangle and the other leg extended through the base, the line joining the points obtained is bisected by the base.

170. A line from a point on a side of an equilateral triangle to a point twice as far from the common vertex on another side is perpendicular to one of the sides of the triangle.

171. A line from the vertex of the right angle of a right triangle to the middle of the hypotenuse divides the triangle into two isosceles triangles.

172. The bisectors of the two interior adjacent angles formed by a transversal with one of two parallels cut off equal sects from the transversal on the other parallel.

173. If two consecutive sides of a parallelogram are extended, not from the common vertex, their own lengths, the line joining their extremities passes through the fourth vertex.

174. One angle of a triangle is double another; the included side is constant, but the angles change. Find the locus of the intersection of the bisector of the larger angle with the opposite side.

175. If the circumcenters of the triangles at the vertices, formed by joining the midpoints of the sides of a triangle, are joined to the midpoints of the two sides which help form each triangle, an equilateral hexagon (not always convex) is formed.

176. In triangle ABC, if YX joins the midpoints of the sides BC and CA, the incenters I and I', of triangles ABC and XCY, and C are collinear, and $CI' = II'$.

177. In the figure of *176*, if S and S' are the circumcenters, $S'I'$ is parallel to SI, and equals one half SI.

NOTE. Compare Exercises *109, 110, 112, 113, 176, 177,* and note that the facts proved of each corresponding pair of points are the same. Upon what one fact about the figure do all these proofs depend?

BOOK II. CIRCLES

SECTION I. DEFINITIONS

160. Circles. The limited portion of a plane bounded by a closed line, every point of which is equidistant from a point within called the *center*, is a *circle*. The closed line that bounds a circle is called its *circumference;* the term "circumference" is often used for the *length of the circumference*, the context usually showing which meaning is to be used. The amount of surface in a circle is called the *area* of the circle. In the advanced subjects of Mathematics, the word "circle" is used for the circumference, and some books on Geometry are now using "circle" in that sense. The meaning is usually clear, and the pupil should not be confused by either use. Some of the exercises will be worded in each way, although in the text the definitions will be adhered to.

**A circumference is a curve.*

For if any part were straight, it could not have all its points equidistant from the center (§ 102).

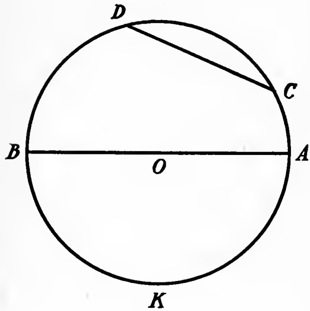
**A circle has but one center.*

For if it had two, they would both be midpoints of the sect through them with its ends on the circumference.

A circle is named by the letter at its center, unless that would cause ambiguity; as, $\odot o$ would mean the circle having its center at o.

A part of a circumference is called an *arc*.

161. Radii, Diameters, Chords. A line from the center of a circle to any point on the circumference is called a *radius*.



A line joining two points of a circumference is called a *chord;* a chord through the center is called a *diameter.*

OB is a radius, *AB* is a diameter, *CD* is a chord.

The parts of the circumference, *BKA*, *ACDB*, are *arcs*, and are named, in the direction contrary to that taken by the hands of a clock, *BA*, *AB*.

**Radii of the same circle are equal* (§ 160).

**Diameters of the same circle are equal.*

162. A point is *within a circle*, *on the circumference*, or *outside the circle*, according as its distance from the center is less than, equal to, or greater than, a radius.

163. Equality of Circles. If two circles have equal radii, they are congruent, for if superimposed they coincide. If two circles are equivalent, they are congruent, for if, when superimposed with the centers coinciding, any point of the circumference of one fell within the other, all points of that circumference would fall within the other, and the circle would be a part of the other, and therefore smaller.

On this account, *equal* will be used for circles that are equivalent, or congruent, or have equal radii, for there is no distinction between these cases.

From the foregoing it is evident that *but one circumference can be drawn with a given center and a given radius.*

A circle is sometimes supposed to be rotated about its center, so that any particular point of the circle takes a different position. The circle as a whole would evidently be in the same position as before rotation, but chords, arcs, radii, etc., would have changed position. This can be used to superimpose one figure in the circle on the place occupied by another figure. (In this case, one figure would be supposed to rotate, while the other would remain in the same position.)

***164.** *Through three points not in the same straight line, one, and but one circumference can be drawn* (§ 152). Points in the same straight line are called *collinear.*

165. Inscribed and Circumscribed Figures. A circumference that passes through the vertices of a polygon is said to be *circumscribed* about the polygon, and the polygon is said to be *inscribed* in the circle.

A circumference that lies within a polygon and touches each of its sides in one point only is said to be *inscribed* in the polygon, and the polygon is said to be *circumscribed* about the circle.

§ 164 might have been stated *One, and but one, circumference can be circumscribed about a given triangle.*

166. *One, and but one, circumference can be inscribed in a given triangle.*

See § 154. Note (by § 155) that three circumferences can be drawn outside a triangle, touching the sides each in one point. Such circumferences are said to be *escribed* to the triangle.

167. Intersection of Lines and Circumferences. A line that meets a circumference at one point only is called a *tangent* to the circle; one that meets the circumference at two points is called a *secant.* The point where a tangent meets a circumference is called the *point of tangency,* or *point of contact.*

(1) **A line perpendicular to a radius at its intersection with the circumference is a tangent.* For all other points on the line are at a distance from the center greater than the radius.

(2) ** A line which meets a circumference at a point, and is not perpendicular to the radius to that point, is a secant.*

For a perpendicular can be drawn to it, and a second oblique can be found equal to the radius, thus showing a second point of intersection with the circle (§ 113).

From the closed-figure intersection axiom, and the preceding statement, it follows that (3) *a line at less than a radius distance from the center of a circle — that is, a line through a point within a circle — is a secant.*

(4) **A tangent to a circle is perpendicular to the radius drawn to the point of contact.* Follows from (2).

(5) **The perpendicular to a tangent at the point of contact passes through the center of the circle.*

(6) **A line at more than a radius distance from the center of a circle does not meet the circumference.*

168. Center Lines. The line through the centers of two circles is called their *center line;* that part of it between the centers is called their *center sect.*

169. Length of the center sect dependent on the position of the two circles in regard to each other.

(1) **If two circumferences do not meet, their center sect is greater than the sum, or less than the difference, of the radii, according as the center of one is outside, or inside, the other.*

(2) **If two circumferences meet on their center line, the center sect equals the sum, or the difference, of the radii, according as the center of one circle is outside, or inside, the other.*

(3) **If two circumferences meet at a point not on their center line, the center sect is less than the sum, and greater than the difference, of the radii.*

If the figures are drawn, the reasons for the statements will be found to depend upon the inequality axiom, and the length relations between the sum or difference of two sides of a triangle and the third side.

170. Intersection of Two Circumferences. (Converses of § 169.)

**If the center sect of two circles*

(1) *is greater than the sum of the radii, the circumferences do not meet, and lie outside each other.*

(2) *is less than the difference of the radii, the circumferences do not meet, and one circle is within the other.*

(3) *equals the sum of the radii, the circumferences meet on the center line, and are outside of each other.*

(4) *equals the difference of the radii, the circumferences meet on the center line, and one circle is inside the other.*

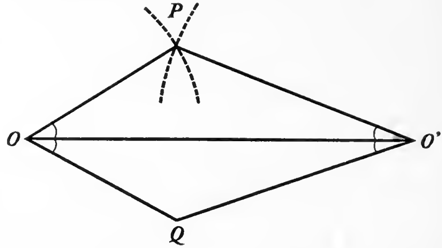
(5) *is less than the sum and greater than the difference of the radii, the circumferences meet at a point not on the center line.*

These statements are used in all cases where it is necessary to show whether or not two circumferences meet.

Note. There are five values for the center sect: less than the difference of the radii, equal to that difference, between the difference and the sum, equal to the sum, and greater than the sum. The pupil should understand the results from each of these lengths.

171. **Two circumferences that meet at a point not on their center line, intersect twice.*

If the circumferences of ⊙s O and O' meet at P, a second triangle $OO'Q$ could be constructed congruent to $OO'P$, by drawing angles on the other side of OO' equal respectively to $\angle O'OP$ and $\angle OO'P$. Why does this show that Q is a second point of intersection?



NOTE. To show that two circumferences intersect twice it is necessary to show.

(1) the center sect less than the sum of the radii.

(2) the center sect greater than the difference of the radii.

172. Common Chord. If two circumferences meet twice, the line joining the points of intersection of the circles is called their *common chord.*

**If two circumferences intersect twice, their center line is the perpendicular bisector of their common chord.* Use equidistance.

173. Tangency of Circles. Two circumferences that meet at one point only are said to be *tangent* to each other, or to *touch* each other. The point where they meet is called the *point of tangency*, or the *point of contact.* The circles also are sometimes spoken of as tangent to each other.

**If two circumferences meet at a point on their center line, they are tangent to each other.* If they meet at a second point on the center line, they are identical; if at a second point which is not on the center line, the center sect would have two different lengths, which is impossible [§ 169 (2), (3)].

**Two circles tangent to the same line at the same point are tangent to each other.*

174. *The shortest sect from a given point to a circumference is a sect of the line from that point through the center.* Why? Prove it when the point is inside and when it is outside.

175. ORAL AND REVIEW QUESTIONS

If the radii of two circles are 10 ft. and 6 ft. long, tell what is known about the position of the circles if the center sect is (1) 12 ft. long, (2) 16 ft. long, (3) 4 ft. long, (4) 2 ft. long, (5) 20 ft. long. Why is a circumference a curve? Why has it but one center? How many points not in a straight line determine a circumference? Can a circumference always be drawn through four points? Can it always be drawn tangent to three lines? When are circles tangent? When is a line tangent to a circle? What is known about the center line of two intersecting circumferences and their common chord? What relation is there between the distance of a line from the center of a circle and the intersection of the line and circumference. If it is necessary to draw two circles which are tangent externally (*i.e.* outside of each other), how long must the center sect be made? If it is necessary to have their circumferences meet twice?

SECTION II. CONSTRUCTIONS

176. Postulates. In plane geometry, the use of two instruments for construction is allowed, or *postulated.* These instruments are the *straight edge* (or *ruler, except that no measurements may be taken with it*) and the *compass.*

The postulates that allow the use of these instruments are:

(1) *A straight line may be drawn from any one point to any other point.*

(2) *A sect may be produced to any length in that line.*

(3) *A circumference may be described with any point as a center and any sect as a radius.*

177. Constructions. In the theorems of Book I, auxiliary lines have been added to the given figure, but only as representations of existing lines about which it was necessary to reason in order to establish the proof. The question of the accuracy of the drawing of these lines had no bearing on the truth of the theorem, for the reasoning was entirely about the figures that the lines were taken to represent. In the problems that are to be done, the proof consists in showing that the figure constructed is—within the limits of accuracy of the instruments used—a correctly drawn figure according to the requirements of the proposition. The theorem *proves a fact;* the construction *makes a required figure,* then proves the correctness of the method of construction.

178. Analysis of a Construction. To discover the method of drawing a required construction, first draw the completed figure as accurately as possible, *without actually constructing it.* By the study of the completed figure, attempt to find what lines must be constructed in order to make the figure in such a way that it can be shown to be the required one. This is really working backwards from the completed figure in an attempt to find upon what it is based. Having analyzed the figure, draw the lines found necessary, and so build up a figure that can be proved to be correct.

Do not neglect the classification method, for it will make the analysis method unnecessary in many cases. For example, a perpendicular can be obtained by finding two points equidistant from the ends of the sect to which the line is to be perpendicular.

179. Construction Uses of the Circumference. The circumference is the *locus of points at a certain fixed distance from a fixed point.* The definition shows this fact, and it is of great importance, both in proofs, and in construction work. The two following hints will show common uses of the circumference.

To draw a line of a given length from a given point, use the compass to draw a circumference around the given point, with the given sect as a radius. This is called *describing* a circumference with a certain center and radius.

To construct a point equidistant from two given points, use each of the given points as a center, and use the same radius. What relation must hold between the center sect and the radii?

178. Construct an equilateral triangle on a given sect as a side.

179. Construct a triangle having its sides equal to three given sects. Is it always possible?

180. Construct an angle equal to two thirds of a right angle.

181. Construct an isosceles triangle on a given base, having each leg twice the length of the base.

182. Construct a regular hexagon on a given sect as a side.

183. Construct three equal circles, each tangent to the other two.

184. Construct seven equal circles, six of them surrounding the seventh and each tangent to three others.

185. Construct two sects, given their sum and their difference.

186. Construct a sect, given a second sect and the sum of the two sects.

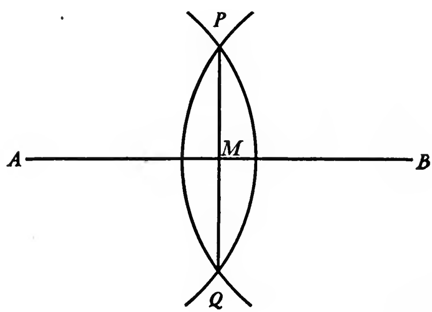
187. Construct a sect, given a second sect and the difference of the two sects.

180. CONSTRUCTION I. *To draw the perpendicular bisector of a given sect.*

ANALYSIS

Two points equidistant from the ends of the given sect will determine the perpendicular bisector.

PROOF



Given. AB.

Required. To draw the $\perp$ bisector of AB.

Construction. I. With center A, and radius AB (or any radius $> \frac{1}{2} AB$) describe a circumference. With center B and the same radius describe a second circumference (circum. post.).

II. These circumferences meet at two points. (Center sect is less than the sum, but greater than the difference of the radii.) Call the points of intersection P and Q.

III. Draw PQ (line post.).

To prove. $PQ \perp AB$ at its midpoint.

Proof. I. $PA = PB$, $QA = QB$ (radii of equal circles).

II. $\therefore$ P and Q are equidistant from the ends of AB, and $PQ \perp AB$ at its midpoint M. (A line through two points equidistant from the ends of a sect is the perpendicular bisector of that sect.)

188. Find by construction the point equidistant from three given non-collinear points.

189. Describe a circumference having its center in a given circle, and passing through two given points.

190. Draw a line through a given point, so that it will be equidistant from two other given points.

191. Construct an isosceles triangle, given the base and the length of the altitude.

181. CONST. II. *To draw a perpendicular to a given line through a given point*

(1) *in the line ;*

(2) *outside the line.*

Use equidistance.

192. Construct an isosceles triangle, given the altitude and a leg.

193. Construct a triangle, given

(1) Two sides and the altitude to the third side.

(2) The base, one of the sects cut off on the base by the altitude, another side.

(3) The base, its median, and its altitude.

194. Construct a rectangle, given a side and a diagonal.

195. Given an angle and a point, draw a line through the point, so as to make equal angles with the arms of the given angle.

196. Construct the point in a given line, such that lines from two given points on the same side of the given line to the point found may make equal angles with the given line.

182. CONST. III. *To draw a perpendicular to a given sect at the end, without extending the sect.*

Let the sect be a leg of the required right triangle, and use the fact known about the midpoint of the hypotenuse.

183. CONST. IV. *To bisect a given angle.*

Draw the lines so that the two parts of the angle may be proved to be equal ; that is, use *angles equal.*

197. Construct the points that are equidistant from three lines that are neither concurrent nor all parallel.

198. Construct an angle equal to one half a right angle ; to one third a right angle.

NOTE. It is impossible to trisect any given angle with no instruments except the compass and ruler. Such exercises as the second part of *198* must be done by using some angle that is known to be one third of a right angle, or that can be made into one third of a right angle by bisection.

199. Construct a point in one side of a given triangle equidistant from the other sides.

184. CONST. V. *At a given point in a given line, to construct a line making a given angle with the given line.*

What is the simplest way to prove two angles in different places equal?

200. Construct a triangle, given two sides and the included angle.

201. Construct a triangle, given two angles and the included side.

202. Construct a triangle, given two angles and any side.

203. Construct an isosceles triangle, given the altitude and the vertex angle.

204. Construct an isosceles triangle, given the altitude and a base angle.

205. Construct a right triangle, given the perimeter and either leg.

185. CONST. VI. *Through a given point to draw a parallel to a given line.*

206. From a point outside a given line, draw a line making a given angle with that line.

207. Construct a triangle, given an altitude and two angles.

208. Construct a triangle, given an angle, one of the sides including the angle, the median from the same vertex. (Use the midpoint of the given side.)

209. Draw a sect terminated by the arms of a given angle, and through a given point within the angle, so that the point will bisect the sect.

210. Draw a line from a given point outside a given angle, through the arms of the angle, so one of the arms will bisect the sect from the point to the other arm.

211. Construct a rhombus, given
(*a*) A side and an altitude.
(*b*) The altitude and a diagonal.
(*c*) The sum of the altitude and a side, an angle.

186. CONST. VII. *To divide a given sect into any number of equal parts.* (Use a system of parallels.)

212. Divide a given sect so that one part of it equals twice the other part; so that one part equals three fifths of the other.

213. Divide a given sect so that the second of three parts equals twice the first, the third equals three times the second.

214. Divide a given sect into four parts, so that the second equals twice the first, the third equals three times the first, the fourth equals four times the first.

187. Sectors and Segments. A *sector* of a circle is that surface bounded by two radii and the arc between their ends. Two radii evidently divide the circle into two sectors. The order of lettering (reverse to the direction taken by the hands of a clock) shows which sector is meant.

A *segment* of a circle is that surface bounded by a chord and the arc between its ends. A chord evidently divides the circle into two segments. Here again, the order of lettering shows which one is meant.

188. Angles in a Circle. A *central angle* is the angle between two radii. Any two radii form two central angles, which are distinguished by the order of their lettering.

An *inscribed angle* is an angle between two chords that meet on the circumference.

The arc that lies between the arms of an inscribed angle or a central angle is said to be *subtended* by the angle, and the angle is said to *stand on* the arc. An arc is also said to be subtended by its chord.

SECTION III. CIRCLE THEOREMS

189. Theorem I. *In the same circle, or in equal circles, if two central angles are equal, the arcs on which they stand are equal also; and of two unequal angles, the greater angle stands on the greater arc.*

190. Cor. 1. *In the same circle, or in equal circles, if two arcs are equal, the central angles standing on those arcs are equal also; and of two unequal arcs, the greater has the greater central angle standing on it.*

191. Cor. 2. *In the same circle, or in equal circles, sectors having equal central angles are equal; and of two sectors having unequal central angles, that which has the greater central angle is the greater.*

192. Cor. 3. *A diameter bisects the circle and its circumference.*

215. An arc is one quarter of a circumference if its central angle is a right angle, and conversely.

Note. One quarter of a circumference is called a *quadrant of arc.*

216. An arc is greater than, or less than, a quadrant, if its central angle is greater than, or less than, a right angle.

193. Semicircle; Major and Minor Arcs; Complements, Supplements, Explements. One half a circle is called a *semicircle*, and one half a circumference is called a *semicircumference.* An arc less than a semicircumference is called a *minor* arc; one greater than a semicircumference is called a *major* arc.

Two arcs whose sum is a quadrant are called *complements* of each other; two whose sum is a semicircumference are called *supplements* of each other; two whose sum is a circumference are called *explements* of each other.

Notice that the part of the surface of a circle cut off by a diameter (a semicircle) is a segment, and also a sector.

217. If two arcs are equal, their complements are equal; their supplements are equal; their explements are equal.

194. Theorem II. *In the same circle, or in equal circles, if two arcs are equal, they are subtended by equal chords; and of two unequal minor arcs, the greater is subtended by the longer chord.*

195. Cor. 1. *In the same circle, or in equal circles, if two chords are equal, they subtend equal minor and equal major arcs; and of two unequal chords, the longer subtends the greater minor arc.*

218. If two equal chords of a circle intersect, they include a pair of equal arcs.

219. An inscribed equilateral polygon is regular.

220. An inscribed equilateral hexagon has each side equal to a radius.

221. If a chord equals a radius, its arc is one sixth of the circumference.

196. Theorem III. *A diameter that is perpendicular to a chord bisects the chord and its subtended arcs.*

197. Cor. 1. *A diameter that bisects a chord is perpendicular to the chord.*

198. Cor. 2. *The perpendicular bisector of a chord passes through the center of the circle.*

222. The perpendicular from the center of a circle to a side of an inscribed equilateral triangle equals one half the radius.

223. The sects of any line intercepted between the circumferences of two circles that have the same center (*concentric circles*) are equal.

224. Find the locus of the center of a circumference that passes through two given points.

225. Find the chord through a given point in a circle, and bisected by that point.

226. Find the locus of the midpoints of parallel chords.

199. Theorem IV. *In the same circle, or in equal circles, equal chords are equidistant from the center; and of two unequal chords, the longer is nearer the center.*

If the chords are placed so that they diverge from a common point on the circumference, and the line joining their centers is drawn, the proof can be easily found. Why is this position a good one in which to place the chords?

200. Cor. 1. *In the same circle, or in equal circles, chords that are equidistant from the center are equal; and of two chords that are unequal distances from the center, the one nearer the center is the longer.*

201. Cor. 2. *A diameter of a circle is greater than any other chord.*

227. If two sects from a given point to a given circumference are equal, those lines are equidistant from the center of the circle; and if the lines intersect the circumference in two points each, the sects from the given point to the second points on the circumference are also equal.

228. If two equal chords intersect, the sects of one equal the sects of the other.

229. Find the locus of the midpoints of the equal chords of a circle.

230. If two equal chords meet at a point on a circumference, the bisector of the angle formed by them is a diameter.

231. If two circles are concentric, all chords of the larger which are tangent to the smaller are equal.

202. Ratio, Proportion. The *ratio* of one magnitude to another like magnitude is that number by which the second magnitude must be multiplied to give the first magnitude. For example, the ratio of a line two inches long to one four inches long is one half; the ratio of a line four feet long to one two feet long is two. A ratio is evidently a quotient.

If the ratio of one pair of magnitudes is equal to the ratio of a second pair of magnitudes, the four magnitudes are said to form a *proportion*, and are said to be *proportional*. The four magnitudes used in a proportion are called *terms*.

WARNING. *It needs* ***four*** *magnitudes to form a proportion. Do not say that two things are proportional.*

203. Measures: Commensurable, Incommensurable. The measure of a magnitude is its ratio to a unit of the same kind; as, the measure of a line is its ratio to an inch, a foot, or some other unit of length. The numerical measure of a magnitude is its ratio to a unit, expressed numerically; as, five feet.

Two magnitudes are *commensurable*, if some third magnitude is contained a whole number of times in each, with no remainder.

Two magnitudes are *incommensurable*, if no third magnitude is contained evenly in both.

The ratio of two like commensurable magnitudes is the ratio of the numbers of times some third magnitude is contained in them; as, the ratio of a line six feet long to one eleven feet long is six to eleven.

NOTE. Incommensurable quantities will be treated in the Appendix, § 348.

204. Theorem V. *In the same circle, or in equal circles, central angles are proportional to the arcs on which they stand.*

The commensurable part of this theorem is all that is required at this time; that is, the pupil may assume that the central angles are commensurable (have a common divisor), and so show that the ratio obtained for them will be the same as the ratio obtained for their arcs. Instead of using numbers to show how many times the assumed common divisor will go into the angles, letters should be used, so that the proof will be general. For the incommensurable discussion, see Appendix, § 348.

205. Cor. 1. *In the same circle, or in equal circles, sectors are proportional to their central angles* (com. case only).

206. Measurement of Central Angles. As a central angle is shown by Theorem V to be the same part of a perigon that its arc is of a circle, it is customary to say that a central angle is *measured by its arc.*

The reason for this can be shown more clearly by the following explanation of the common method of measuring angles:

If a circumference is divided into 360 equal parts called *arc degrees*, the radii joining the points of division to the center will divide the perigon at the center into 360 equal parts called *degrees of angle;* also, if each arc degree is divided into 60 equal parts, called *minutes*, and each arc minute into 60 equal parts, called *seconds*, the radii to these points of division will form *minutes*, and *seconds*, *of angle*. Then if any arc can be expressed in degrees, minutes, and seconds, its central angle will evidently be composed of the *same numbers* of degrees, minutes, and seconds of arc; that is, the angle is measured by its arc.

Theorem V might then have been stated: *A central angle is measured by its arc.*

207. Theorem VI. *An angle inscribed in a circle is measured by one half its arc.*

To make this theorem complete, it must be proved for an inscribed angle having

(1) one arm a diameter;

(2) one arm each side of the center;

(3) both arms the same side of the center.

Upon what must this theorem depend?

208. COR. 1. *In the same circle, or in equal circles, inscribed angles on equal arcs are equal; and of two inscribed angles on unequal arcs, that on the larger arc is the larger angle, and conversely.*

209. COR. 2. (1) *An angle inscribed in a semicircle is a right angle.* (2) *An angle inscribed on an arc greater than, or less than, a semicircumference, is greater than, or less than, a right angle.*

232. What would be the locus of the vertices of all triangles having a fixed base and a constant vertex angle?

233. Prove that the bisectors of the vertex angles of all triangles having the same base and equal vertex angles on the same side of that base, are concurrent.

234. If a right triangle is inscribed in a circle, its hypotenuse is the diameter of the circle.

235. A circumference described on any side of a triangle as a diameter passes through the feet of the altitudes to the other sides. In an isosceles triangle, it bisects the base if a leg is used as diameter.

236. A circumference described on the hypotenuse of a right triangle as a diameter passes through the vertex of the right angle.

237. Circumferences described on two sides of a triangle as diameters intersect on the third side.

238. Three consecutive sides of an inscribed quadrilateral subtend arcs of 68°, 97°, 59°, respectively. Find each angle of the quadrilateral, and the angles between the diagonals.

239. Two angles of an inscribed triangle are 75°, 95°. Find the arcs subtended by the sides.

210. Theorem VII. *An angle formed by a tangent and a chord of a circle is measured by one half the arc subtended by the chord.* Notice that there are two angles between the chord and the tangent, and that each is measured by one half the arc on the same side of the chord as the angle measured.

211. Cor. 1. *Tangents from the same point to a circumference are equal.*

212. Cor. 2. *The line joining the point of intersection of two tangents to the center of the circle to which they are tangent bisects the angles between the tangents, and the central angle between the radii drawn to the points of contact.*

240. Tangents from a point a radius distant from a circle make with their chord of contact an equilateral triangle.

241. The sum of two opposite sides of a circumscribed quadrilateral equals the sum of the other two sides.

242. A parallelogram circumscribed about a circle is a rhombus.

243. If the vertices of a circumscribed quadrilateral are joined to the center of the circle, the non-adjacent central angles formed are supplemental.

244. The angle between two tangents to a circle is double the angle between the chord of contact and the radius to a point of contact.

245. If two circles are tangent to each other, the common tangent at their point of tangency bisects the other common tangents.

213. Theorem VIII. *If two parallel lines meet a circumference, the arcs cut off between them are equal.*

246. Lines through the points of intersection of two circles, parallel to each other, are equal.

247. If the ends of two unequal parallel chords are all joined, the points of intersection and the center of the circle are collinear.

248. If the ends of two equal arcs of a circle are all joined, two of the lines are parallel, the other two cut off equal sects on each other.

214. Theorem IX. *An angle formed by two chords that intersect within a circle is measured by one half the sum of the arcs intercepted by the arms of the angle.*

215. Theorem X. *An angle formed by two secants, two tangents, or a secant and a tangent, that intersect outside the circle, is measured by one half the difference of the intercepted arcs.*

Note. To complete the discussion of angles whose arms, or arms extended, meet a circumference, the measurement of the supplements of the angles mentioned in *207*, *210*, *214*, and *215* should be investigated.

249. An exterior angle of an inscribed quadrilateral equals the opposite interior angle.

250. The sum of the angles inscribed in the segments of a circle cut off by the sides of an inscribed quadrilateral equals three straight angles.

251. The ends of two arcs of 120° and 60° are joined. How large are the angles formed? (Two cases.)

252. The opposite sides of an inscribed quadrilateral, no two of whose sides are parallel, are extended until they meet. Find the angle between the bisectors of the angles formed.

253. If two chords do not intersect, and the midpoint of one minor arc of one chord is joined to the ends of the other chord, the two triangles formed are mutually equiangular (that is, the angles of one are equal respectively to the angles of the other).

254. If three angles have their arms pass through the ends of the same chord of a circle, their vertices being on the same side of the chord, one inside the circle, one on the circle, the other outside the circle, the angle whose vertex is outside is the largest, the one whose vertex is inside is the smallest.

216. Theorem XI. *The opposite angles of an inscribed quadrilateral are supplemental.*

217. COR. 1. *A parallelogram inscribed in a circle is a rectangle.*

218. COR. 2. *If the opposite angles of a quadrilateral are supplemental, the quadrilateral is inscriptible.*

Through how many vertices can a circumference be drawn? What if the other fell within the circle? outside the circle?

Four or more points that lie on the same circumference are said to be *concyclic.*

255. A rectangle is inscriptible, and its diagonals intersect at the center.

256. In an inscribed hexagon, the sum of three angles, no two adjacent, equals the sum of the other three. What general statement for polygons of an even number of sides can be made?

257. A circle described on the line joining the orthocenter of a triangle to a vertex passes through the feet of two altitudes.

SECTION IV. CONSTRUCTIONS DEPENDING ON CIRCLE THEOREMS

219. Const. VIII. *To bisect a given arc.*

258. Bisect an inscribed angle, without using its arms.

259. Draw a tangent to a given arc at a given point, without using the center of the circle.

220. Const. IX. *To find the center of a circle, given any arc.*

260. Find the center of a circle, given the position of a chord and the length of the diameter.

261. Describe a circle, given the positions of two chords.

221. Const. X. *To draw a tangent from a given point to a given circle.*

262. Draw a chord of a given circle, through a given point, so that it will have a given length.

263. Draw a line through a given point, so that its distance from a given point may equal a given sect.

264. Construct a triangle, given the base, and the altitudes to the other sides.

265. Construct a triangle, given the base, the altitude to the base, and a second altitude.

222. Const. XI. *To construct a circumference, so that a given angle will be inscribed on the arc subtended by a given chord.*

What other kind of an angle is measured by the same arc as an inscribed angle? Can this angle be constructed, given the chord in a fixed position? How can the center then be found?

This theorem is the foundation of many exercises on loci. The most common way of using this as an example of locus is: "*Find the locus of the vertex of a given angle opposite a given fixed sect.*"

When a locus gives a circumference for its result, it is because of one of two things:

(1) because the point in question is always at a fixed distance from a given fixed point;

(2) because a given angle is always opposite a given fixed sect, the most common and most important case being that of a right angle opposite the fixed sect, when the locus is the circumference on the sect as diameter.

266. Construct a triangle, given the base, the vertex angle, the median to the base.

267. Construct a triangle, given the base, the vertex angle, the altitude.

268. Construct a triangle, given the base, another side, the angle between the median to the base, and the third side.

269. Construct a triangle, given the base, the altitude to the base, the angle between the median to the base, and a side.

270. Find the locus of the vertex of a triangle, given the base and the vertex angle; the base and the angle between the median and a side.

271. Find the locus of the midpoints of all chords whose lines pass through a given fixed point.

272. Find the locus of the midpoint of a sect of fixed length, whose ends are on the arms of a given fixed right angle.

273. Construct a parallelogram, given a diagonal, an angle opposite, the angle between the diagonals.

274. Construct a trapezoid, given the bases, a side, and the angle between the diagonals.

223. **SUMMARY OF THEOREMS AND COROLLARIES, BOOK II**

(Numbers in parentheses refer to black-faced section numbers.)

I. Circumferences Meet Twice. c. s. $<$ sum radii and $>$ diff. (170).

II. Circles Tangent. c. s. $=$ sum radii or $=$ diff. (170); tangent same line (173).

III. Line Meets a Circumference Twice. Not $\perp$ radius (167).

IV. Line Tangent to a Circle. $\perp$ radius at end (167).

V. Lines Equal. Center line $\perp$ bisector common chord (172); chords of equal arcs (194); diameter $\perp$ chord (196); $=$ chords equidistant from center, and conversely (199, 200); tangents (211).

VI. Lines Unequal. Chords of unequal arcs (194); $\neq$ chords unequal distances from the center, and conversely (199, 200); diameter greatest chord (201).

VII. Lines Perpendicular. Diameter bisecting chord (197); ▱ insc. in circle (217).

VIII. Line Passes through Center. $\perp$ bisector chord (198); $\perp$ tangent at point of tangency (167).

IX. Angles Equal. Central angles having equal arcs (190); inscribed angles on equal arcs (208); angles bisected by line from center to intersection of tangents (212).

X. Angles Unequal. Central angles on unequal arcs (190); inscribed angles on unequal arcs (208, 209).

XI. Angles Supplementary. Opp. angles insc. quad. (216).

XII. Angles Measured. Central angles, by their arcs (204, 206).

(*a*) Angles with vertices outside the circle, by one half the difference of the intercepted arcs (215).

(*b*) Angles with vertices on the circumference, including inscribed and tangent and chord angles, by one half the intercepted arc: an angle on a semicircumference equals a right angle; one on more, or less than, a semicircumference, is obtuse or acute (207, 210, 209).

(*c*) Angles with vertices inside the circle, by one half the sum of the intercepted arcs (214).

XIII. Arcs Equal. Equal chords (195); equal central angles (189); $\perp$ bisector chord bisects arc (196); diam. bisects circumference (192); equal insc. angles (208); between parallel lines (213).

XIV. ARCS UNEQUAL. Unequal central angles (189); unequal chords (195); unequal inscribed angles (208).

XV. SECTORS EQUAL. Equal central angles (191); diameter bisects circle (192).

XVI. SECTORS UNEQUAL. Unequal central angles (191).

XVII. SECTORS PROPORTIONAL. To their central angles (205).

XVIII. QUADRILATERAL INSCRIPTIBLE. Opp. angles sup. (218).

CONSTRUCTIONS

XIX. LINE PERPENDICULAR.

(*a*) Bisector (180).

(*b*) At point in the line (181).

(*c*) From a point to a line (181).

(*d*) At the end of the sect, without extending (182).

XX. BISECT ANGLE (183).

XXI. ANGLE EQUAL TO A GIVEN ANGLE (184).

XXII. PARALLEL TO A GIVEN LINE (185).

XXIII. DIVIDE SECT INTO EQUAL PARTS (186).

XXIV. BISECT ARC (219).

XXV. FIND CENTER OF A GIVEN CIRCLE (220).

XXVI. TANGENT (221).

XXVII. CIRCUMFERENCE HAVING A GIVEN ANGLE OPPOSITE A GIVEN FIXED SECT (222).

LOCI

XXVIII. POINTS EQUIDISTANT FROM A FIXED POINT (179).

XXIX. VERTEX OF A GIVEN ANGLE OPPOSITE A FIXED SECT (222). (See XXVII.)

224. ORAL AND REVIEW QUESTIONS

Why is the intersection of the perpendicular bisectors of the sides of a triangle called the circumcenter? Why are the intersections of the angle bisectors called the in- and ex-centers? If the center sect of two circles is 10 ft. long, the radii being 8 ft. and 4 ft., what is known about the meeting of circumferences? For what length center sect, with the same radii, would the circles be internally tangent? When is a line tangent to a circle? How can you construct a tangent at a point on a circle? from an external point to a circle? How can you get a right angle opposite a given sect? any angle? Express this as a locus. Of what is a circumference the locus? Define chord. What special kind of chord is there? Which is the longest chord of a circle? Why? When are arcs equal? central angles equal? inscribed angles equal? How can you measure an angle with its vertex inside the circle? its vertex on the circle? its vertex outside the circle? What kinds of angles have their vertices on the circumference? outside the circle? What is known about the angles of an inscribed quadrilateral? How would an exterior angle of the quadrilateral compare with the opposite interior angle? Why? What principle underlies the construction of perpendiculars? Is it also used for the proofs? Why does the perpendicular bisector of a chord pass through the center? Give a second proof. What kind of triangle is always formed by two radii and a chord? Can you tell any proofs where this fact is of use? What is the key to drawing lines parallel? to making equal angles? to dividing a sect into any number of equal parts? How much of a circle is needed to find its center? Are lines joining the ends of equal arcs always parallel? What is known about all equal chords of a circle? What is the locus of the midpoints of all such chords? Upon what two characteristics of a circle do the circle loci depend? How many points need to be fixed in the first? in the second? Explain what is meant by a central angle being measured by its arc. What is the shortest line from a point to a circumference? Why? What relation has the angle between two tangents to the angle between the radii to the points of tangency? What two ways are there of measuring the angle between two tangents? If the arms of a central angle, an inscribed angle, a tangent and chord angle, are all respectively parallel, how do their arcs compare? Where does the bisector of an inscribed angle meet the arc? If two tangents to a circle are parallel, what part of the circumference is between them?

GENERAL EXERCISES

275. If two circles are tangent internally, and the radius of the larger is the diameter of the smaller, any chord of the larger drawn from the point of contact is bisected by the smaller.

276. Construct a rhombus, given the sum of the diagonals and an angle.

277. Construct a right triangle, given the hypotenuse and the sum of the other sides.

278. An inscribed equiangular polygon of odd number of sides is equilateral; of even number of sides, has each side equal to the second from it.

279. The tangents at the vertices of an inscribed quadrilateral, of which two opposite angles are right angles, form a trapezoid.

280. Construct a rectangle, given the difference between two consecutive sides and a diagonal.

281. If two circles intersect twice, the angle formed by joining a point of intersection to the ends of a secant through the other point of intersection is always the same.

282. If the feet of the altitudes of a triangle are joined, the angles of the triangle formed (called the *pedal* triangle) are bisected by the altitudes.

283. Construct a triangle, given the altitude, a base angle, the sect of the base between the foot of the altitude and the second vertex.

284. Construct a triangle, given the base, its median, a base angle.

285. Construct a rectangle, given a diagonal and the angle between the diagonals.

286. Construct a rectangle, given the sum of two consecutive sides, and the angle between a side and a diagonal.

287. A circumscribed equiangular polygon is equilateral.

288. A circumscribed equilateral polygon of odd number of sides is regular, and has its sides bisected by the points of contact.

289. A circumscribed equilateral polygon of even number of sides has its alternate angles equal.

290. A circumscribed equilateral triangle has each altitude equal to three radii.

291. Construct an isosceles trapezoid, given the longer base, the sum of altitude and side, an angle.

292. Construct a parallelogram, given the sum of two consecutive sides, an angle, and a diagonal.

293. Construct the common tangent to two given circles. (Draw the tangent, the radii to the points of contact, see what parts of the trapezoid formed are given, then try to construct it from those given parts. Notice that four common tangents can be drawn in certain cases.)

294. If two circles are internally tangent at P, and the chord AB of the larger circle is tangent to the smaller circle at T, prove that PT bisects the angle APB.

295. If triangle ABC is circumscribed about a circle O, with CA and AB tangent at fixed points T and T', and BC any third tangent, then $CA + AB - BC$ is constant, and equals the diameter if $\angle A$ is a right angle.

296. If two parallel tangents are crossed by a third tangent, the third tangent subtends a right angle at the center.

297. In *295*, if the triangle is escribed, $AB + BC + CA$ is constant.

298. Construct a trapezoid, given the four sides.

299. Construct a trapezoid, given the parallel bases and the diagonals.

300. Construct a square, given the sum of a diagonal and a side.

301. Construct a square, given the difference of a diagonal and a side.

302. If the sum of two opposite sides of a quadrilateral equals the sum of the other two sides, the quadrilateral can be circumscribed about a circle.

303. If two circles intersect in two points, and diameters are drawn from one of those points, the line joining the other ends of the diameters is double the center sect, and passes through the other point of intersection.

304. Through a point within a circle, draw the chord which is bisected by that point, and prove that it is the shortest chord through the point.

305. Describe a circle tangent to two given lines, to one at a given point.

306. Describe a circle through a given point, touching a given line at a given point.

307. Describe a circle tangent to a given line at a given point, and tangent to a given circle. (Two cases.)

308. In a circumscribed cross quadrilateral the difference of two opposite sides equals the difference of the other two opposite sides.

309. Find the locus of the center of a circle which has a given radius, and cuts a given circle at the ends of a movable diameter.

310. Find the locus of the extremities of tangents of a certain length drawn to a given circle.

311. If perpendiculars are drawn from the ends of a moving diameter to a fixed chord, the sum of those perpendiculars is constant.

312. Construct the bisector of an angle without using its vertex.

313. Construct a right triangle, given the two sects into which the bisector of the right angle divides the hypotenuse.

314. Construct a right triangle having a given fixed hypotenuse, and having the vertex of the right angle in a given fixed line.

315. Construct a right triangle, having a fixed hypotenuse, and having its vertex at a given distance from a given point.

316. Construct a right triangle, having a fixed hypotenuse, and having its vertex at a given distance from a given line.

317. Draw a circle with a given radius, tangent to two given circles.

318. Construct a right triangle, given the radius of the inscribed circle and the altitude to the hypotenuse.

319. Draw a line at a given distance from two given points.

320. Construct a right triangle, given the radius of the inscribed circle and the bisector of the right angle.

321. Construct a right triangle, given the radius of the circumscribed circle, and (a) an acute angle, (b) the difference of the acute angles.

322. Construct a triangle, given the radius of the circumscribed circle, and (a) two sides, (b) two angles, (c) the sects of the base made by the altitude.

323. Describe a circle of given radius, tangent to two given lines.

324. Describe a circle tangent to two given lines at a given distance from their intersection.

325. Draw a right triangle, given the altitude and the bisector of the right angle.

326. Construct a triangle, given a side, the bisector of the vertex angle, the sect of the base between the foot of the altitude and the end of the bisector of the vertex angle.

327. Construct a triangle, given the sum of the sides, two angles.

328. Construct a triangle, given the sum of two sides, and two angles.

329. Construct a triangle, given the sum of two sides, one of the opposite angles, the altitude to the third side.

330. Construct a triangle, given the altitude to the base, a base angle, and the sect from that vertex to the foot of the bisector of the vertex angle.

331. Draw a circle on one side of a triangle as a diameter, and so show that the altitudes from the ends of that side make equal angles with the other sides to which they are not perpendicular.

332. If two circles are tangent externally, and two secants are drawn through their point of tangency, the chords joining the ends of the secants are parallel.

Note. There are many other triangle constructions of the same kind as those in the preceding set, and the pupil can make exercises for himself, by taking any three parts which are independent (that is, such that no one could be obtained from the other two) and trying to form the triangle from those parts. Some of the triangle constructions cannot be done until the fourth book has been studied.

Notice that when two circles are used, the center line is almost always necessary to the figure, and that if the two circles intersect, or are tangent, the common chord, or the common tangent at their point of tangency, is very likely to be needed.

BOOK III. EQUIVALENCE AND AREA

SECTION I. DEFINITIONS AND THEIR DISCUSSION; FORMULAS

225. Equivalence. Two closed figures have been defined as being equivalent when they contain the same amount of surface (§ 34). The question at once arises as to what ways of proving figures equivalent are known. The distinction between the different kinds of equality, as shown in §§ 32–37, and the equality axioms (§ 38), give the foundation for this class; the two following methods show how these definitions and axioms can be used.

(1) Congruent figures are equivalent, and equivalent figures added, subtracted, and multiplied or divided by the same number, give equivalent results (although the results need not be congruent).

(2) The whole equals the sum of all its parts, and the other equality axioms can be applied to the equation obtained.

The equivalence class must then depend on these two methods: the equivalence of two figures usually on the first, equivalence equations usually on the second. When the first is used, the congruent figures are often added or subtracted in different positions, thus forming new figures which are equivalent, but not congruent.

226. Addition of Polygons. The addition of two polygons is accomplished by placing the polygons entirely outside of each other, as regards their surface, but with some portion of the perimeter (either a point, or some one or more sects) in common, and then taking the whole figure thus formed. The common part of the boundary line is considered as omitted, unless it is a point. It is evident that this is simply the ordinary understanding of a sum, the only difference being that the polygons have to be placed in such a position that their sum can be shown as a single figure.

227. Subtraction of Polygons. The subtraction of two polygons is accomplished by placing the smaller polygon entirely inside the larger, and then omitting the part occupied by the smaller; the remaining part of the larger polygon is the difference. Here again the usual idea of subtraction is employed, the figures being placed so that the difference shows as a new figure.

228. Addition and Subtraction of Parallelograms. From the definitions of addition and subtraction of polygons, it can be seen that two parallelograms which have a side and an angle of one equal to a side and an angle of the other can be added or subtracted so as to give a new parallelogram having the same side, and the same angle, as the result. This applies to rectangles having one equal side, and the fact can be put in formal statement as follows:

Two rectangles having an equal side can be added so as to form a rectangle having the equal side for one of its sides, the sum of the two other length sides of the given rectangles for its other side. Similarly for the difference.

229. Rectangles and Squares. A rectangle having the sides X and Y is spoken of as the $\square X, Y$. This is permissible, since all rectangles having those sides are congruent. Similarly, a square on the side x is spoken of as the □ x.

Remembering that all rectangles which have two sides equal are congruent, it is easily seen that all that it is necessary to do in order to multiply a rectangle, is *to multiply one side*, for that makes a new rectangle, composed of as many congruent rectangles as the number by which the side was multiplied. This is most easily seen by keeping in mind that multiplication is here strictly a kind of addition, and that the line is multiplied by continuing it until it contains the required number of equal parts. In exactly the same way, a rectangle can be divided into any number of equal parts by dividing one side into that number of equal parts, and so forming a new rectangle, having one side the required part of the given side.

NOTE. Dividing both sides of a rectangle, or multiplying both sides, performs the operation on the rectangle *twice*. For example, if each side of a rectangle is doubled, the rectangle is made four times as large.

230. Formulas. By the use of the axiom, "the whole equals the sum of all its parts," it is possible to obtain many equations between rectangles and squares, the following of which are the most important:

*(1) *The square on the sum of two sects is equivalent to the sum of their squares plus twice their rectangle.*

Take two given sects, draw the square on their sum, and see if it contains the required parts.

*(2) *The square on twice a sect is equivalent to four times the square on the sect.*

*(3) *The square on the difference of two sects is equivalent to the sum of their squares, less twice their rectangle.*

Try to add the squares of the sects, and subtract the two rectangles in such a manner that a square will be left.

*(4) *The difference of the squares on two sects equals a rectangle, having one side equal to the sum of the sects, the other side equal to the difference of the sects.*

Subtract the smaller square, then make *one rectangle* out of the remainder.

These geometric formulas, which deal entirely with the surfaces of the figures, correspond very closely to certain algebraic formulas; the only differences between the two are that where "square on a sect" appears in the Geometry, "square of a number" appears in Algebra, and where "rectangle of two sects" appears in Geometry, "product of two numbers" appears in Algebra. This correspondence follows throughout; wherever an algebraic formula is entirely of the second degree, the corresponding geometric statement concerning squares and rectangles is also true. The reason for this correspondence will be evident from the third theorem and its corollaries.

333. Find the formula for the square on the sum of three sects. Try to show what general formula can be obtained.

334. What does the square on three times a sect equal? What general formula is there?

SECTION II. THEOREMS

231. Theorem I. *Parallelograms (or triangles) on the same base, or on equal bases, and between the same parallels, are equivalent.*

Do not add, for this method does not work for all positions of the parallelograms.

232. Cor. 1. *Parallelograms (or triangles) on the same base, or on equal bases, and having equal altitudes, are equivalent.*

Place their bases on the same line, then show what?

335. Triangles having two sides equal, and the included angles supplemental, are equivalent.

336. Any two medians of a triangle form, with the side from whose ends they are drawn, and the halves of the sides to which they are drawn, two equivalent triangles.

337. Any median divides the triangle into two equivalent triangles.

338. The three medians of a triangle divide it into six equivalent triangles.

Note. In the following theorems, it is the surface of the figure which is being considered.

233. Theorem II. *Two rectangles having equal altitudes are proportional to their bases.* (Commensurable case only.)

234. Cor. 1. *Two rectangles having equal bases are proportional to their altitudes.*

235. Theorem III. *The ratio between two rectangles equals the product of the ratio of their bases by the ratio* *of their altitudes.* Can their ratio be found at once, if they have no equal dimension? To what could the ratio of each be found?

In this theorem, and, in fact, in most of the theorems of this book, it is convenient to use a single letter to represent each side of the figure rather than to name it by its end points.

236. Area. The *area* of a figure has been defined as the amount of surface inclosed, and this area is usually stated in terms of some area *unit.* It is customary to call the number of length, or *linear* units in a sect, its *length;* and to call the number of area units in a surface its *area.* The unit of length is any convenient sect, such as a sect one inch long, or one yard long; the unit of area is a square having a linear unit for a side.

In dealing with area, it is necessary to use like units; that is, if the inch is used as a length unit, then the square inch must be used as the area unit, and so for all units.

To find the area of a figure, then, it is necessary to take the ratio of the figure to a square unit; while to find the length of a sect, it is necessary to take its ratio to a linear unit. While the terms "area" and "length" will be used in this way, the student should bear in mind that it is the *number* of units which is being used, both in area and in length.

237. Theorem IV. *The area of a rectangle equals the product of its base by its altitude.* Read § 236 carefully, then take the ratio of the given rectangle to a unit of area.

238. Cor. 1. *The area of any parallelogram equals the product of its base by its altitude; the area of any triangle equals one half the product of its base by its altitude.*

239. Cor. 2. *The area of a trapezoid equals one half the sum of its bases times its altitude.*

339. The area of a trapezoid equals its median (the line joining the midpoints of the legs) times its altitude.

340. The area of a circumscribed polygon equals one half its perimeter times the radius of the circle.

341. A line through the midpoint of the median of a trapezoid, cutting the bases, divides the trapezoid into two equivalent parts.

240. Theorem V. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*

This theorem is the famous *Pythagorean proposition*, so called because it is said to have been discovered by Pythagoras. It is probably the most important theorem in Geometry as far as numerical applications are concerned. It has been claimed that about 350 proofs for this theorem have been discovered; it is certain, at any rate, that the number of proofs is very great, and as they are all applications of the equivalence relations that have been given in this third book, the student ought not to have great difficulty in finding several. The only thing to be avoided is the use of area in its arithmetical sense, for this theorem must be proved in a strictly geometrical sense, and so the area relations (that is, the formulas for the area of rectangles, triangles, etc.) must not be used. Many of the proofs are obtained by adding and subtracting equals, so as to form the square on the hypotenuse in one case, and the sum of the squares on the other sides in the other case. Probably the best of these proofs is obtained by drawing the square on the hypotenuse (so as not to include the triangle) and completing the rectangle through its vertices, using the legs continued as two sides of the constructed rectangle. This new figure can be shown to have two different equivalence values; one is obtained from the fact that it can be proved to be the square on the sum of the legs, the other by the axiom of the whole. These two values will quickly give the required relation between the squares.

241. COR. 1. *The square on one leg of a right triangle is equivalent to the square on the hypotenuse less the square on the other leg.*

242. Numerical Applications. Since the area of a figure is the *number* of units in the figure, many of the equivalence relations can be used for numerical applications. For example, the area of a triangle in square units can be obtained when the lengths of base and altitude are known in length units; as, if the altitude of a triangle is 4 ft. long, its base 10 ft. long, then the area of the triangle is $4 \times 10 \div 2$ (or 20) sq. ft. This fact applied to the right triangle formulas enables one to find any side of a right triangle if the other two sides are known; as, in a right triangle having legs 3 ft. long, and 4 ft. long, respectively, the hypotenuse is 5 ft. long, for the area of a square is evidently the square of the number of units in one side, and $3^2 + 4^2 = 5^2$. Similarly, if the hypotenuse is 13 ft. long, and one leg is 12 ft. long, $13^2 - 12^2 = 169 - 144 = 25 = 5^2$; showing that the other leg must be 5 ft. long.

342. Find the area of a parallelogram of base 12 ft. and altitude 14 ft.; of a triangle having the same base and altitude.

343. In order that a triangle should be equivalent to a parallelogram of the same base, what must be true of their altitudes?

344. Find the area of a trapezoid of bases 10 and 18 ft., altitude 11 ft.

345. If one leg of a right triangle is 3 ft. long, the other 8 ft. long, find the length of the median to the side 8.

346. If two chords of a circle are perpendicular to each other, the sum of the squares on the sects formed is equivalent to the square on the diameter.

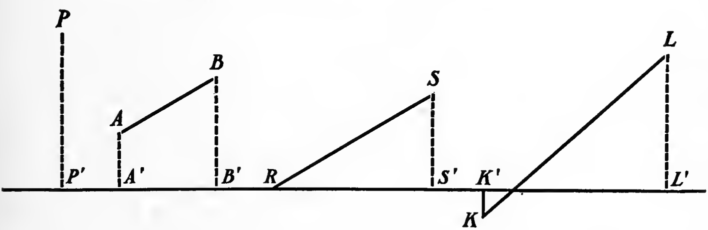
347. In right triangle ABC, C being the right angle, BD is drawn to D on AC. Prove that the square on BD plus the square on AC is equivalent to the square on AB plus the square on CD.

348. In acute triangle ABC, let $AB = 7$, $BC = 8$, $CA = 9$; call the altitude to 7, h, and the sect from A to the foot of the altitude, p. Then two right triangles are formed, and an equation can be written for each. Write these two equations, and solve them for h and p. Find the area of ABC. Draw the median to AB and find its length from the new right triangle formed.

NOTE. This exercise shows that the use of the theorem for the squares on the sides of a right triangle will give, when the lengths of the sides of a triangle are known,

(1) the length of each altitude;
(2) the length of each median;
(3) the lengths of the sects of the sides made by the altitudes;
(4) the area, using the altitude found.

243. Projections. The foot of the perpendicular from a point to a line is called the *projection* of that point on the line. If both ends of a sect are projected on a line, the sect between the projections of the ends of the given sect is called the *projection of the sect on the line.*



The projection of P is P', of AB is $A'B'$, of RS is RS', of KL is $K'L'$.

244. Theorem VI. *The square on a side of an oblique triangle equals the sum of the squares on the other two sides*

(1) *plus twice the rectangle of either of those sides and the projection of the other side on its line, if the opposite angle is obtuse;*

(2) *less twice the rectangle of either of those sides and the projection of the other side on its line, if the opposite angle is acute.*

The acute angle case will have to be proved in both the acute and the obtuse angled triangle. In what kind of a triangle is the square on a side known? Then what must be done?

This theorem is also adaptable to numerical work, and if the letters a, b, c, stand for the sides of a triangle, with a' for the projection of a on b, and b' for the projection of b on a, the formula obtained would be $c^2 = a^2 + b^2 \pm 2\,a'b$ (or $2\,ab'$). Note that if three of these parts are known, the others can be found, and that the kind of angle opposite a side can be told by comparing its square with the sum of the other squares.

349. If one side of a triangle is 5, the projection of this side on a second side is 2, and the other sect of that side is 4, find the third side.

350. Does Th. VI apply to right triangles also? See what the projection would be in the case of a right triangle. Can you make a general statement for the square on any side of any triangle, regarding the projection as negative when it takes off from a side, positive when it adds to the side?

351. In triangle of sides a, b, c, find the lengths of the sects of c made by the altitude to c.

352. Using Th. VI, find the projection of b on c, if $a = 7$, $b = 10$, $c = 13$.

245. Area and Altitude Formulas. It is evident that the altitude and the area of a triangle can be found when the three sides are known, either by the right-triangle method, or by the use of the formulas for the square on a side in an oblique triangle. If the altitude and area are worked out for a triangle of sides a, b, c, a general formula can be obtained, which can be applied very rapidly. If h is the altitude to c, b' the projection of b on c, show that

(1) $b' = \dfrac{\pm(b^2 + c^2 - a^2)}{2c}.$

(2) $h = \sqrt{(b + b')(b - b')}$

$$= \frac{1}{2c}\sqrt{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)},$$

or $\quad h = \dfrac{1}{2c}\sqrt{(a + b + c)(-a+b+c)(a-b+c)(a+b-c)}.$

(3) Calling the sum of the sides $2s$, and substituting,

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Application. If the sides are 7, 8, 9, then s (or half the sum) $= 12$. Subtracting each of the sides, the three remainders, 5, 4, 3, are obtained. Then

$$\text{area} = \sqrt{12 \times 5 \times 4 \times 3} = 12\sqrt{5}.$$

To find any altitude by this method, divide the area by half the side to which the altitude is drawn; for example, the altitude to 5 is $\frac{24}{5}\sqrt{5}$.

246. The Half Equilateral Triangle; the Isosceles Right Triangle. There are two right triangles in which the relative lengths of the sides can be told at once, and these triangles must be used when angles are given to find the lengths of sides, or sides are given to find the size of angles.

If an altitude is drawn in an equilateral triangle, two right triangles are formed, called half equilateral triangles. The altitude meets the opposite side at its midpoint, so one leg is half the hypotenuse; the other leg is found by the squares on the sides. The altitude is always *half the side of the equilateral triangle times the square root of three; and the area of the equilateral triangle is the square of half the side times the square root of three.* If the side is 10, the altitude is $5\sqrt{3}$, and the area is $25\sqrt{3}$. This triangle has angles equal to one third a straight angle, and one sixth a straight angle; or, in degrees, of 60° and 30°.

In an isosceles right triangle, the hypotenuse is a leg times the square root of two. (Why?) The angles here are one half a right angle, or 45°.

353. If the sides of a triangle are 29, 31, 50, find the area and the altitude to 50, by the area formula.

354. How long is the altitude of an equilateral triangle of side 14?

355. If the altitude of an equilateral triangle is $20\sqrt{3}$, find the side.

356. If the area of an equilateral triangle is $49\sqrt{3}$, find the altitude.

357. If the altitude of an equilateral triangle is 10, find the side; if the area is 10.

358. If a leg of an isosceles right triangle is 5, find the hypotenuse.

359. If two thirds of one side of an equilateral triangle, and one third of a second side, are cut off from the same vertex, the line joining those points is perpendicular to the side from which one third is cut off. Does this need to be one third? What general statement can be made?

360. If the vertex angle of an isosceles triangle is 30°, and each leg is 10, find the base; if each leg is *a*.

361. In a triangle having an angle of 120°, show that the square on the opposite side equals the sum of the squares on the other two sides, plus the rectangle of those sides.

362. In a triangle having one angle equal to one third of a straight angle, the square on the opposite side equals the sum of the squares on the other two sides less the rectangle of those sides.

247. Theorem VII. *The sum of the squares on any two sides of a triangle is equivalent to twice the square on one half the third side, plus twice the square on the median to that side.*

The previous theorems allow us to take but one square at a time; therefore, for many purposes this theorem is better, for it takes two squares at once. It has a second purpose, for from it one can find the median in terms of the sides, or one side in terms of the other sides and a median.

248. Cor. 1. *The difference of any two sides of a triangle is equivalent to twice the rectangle of the third side and the projection of its median on it.*

363. Find the median to 5 in the triangle of sides 4, 5, 6.

364. Find the projection of the median to 5 on that line, if the sides of the triangle are 4, 5, 6.

365. If the median to a side of a triangle is 7, the other sides are 8 and 9, find the side.

366. Show that the sum of the squares on the sides of any quadrilateral equals the sum of the squares on the diagonals, plus four times the square on the line joining their midpoints. What does this formula become if the quadrilateral is a parallelogram?

367. The sum of the squares on the diagonals of a trapezoid is equivalent to the sum of the squares on the legs, plus twice the rectangle of the bases.

368. In the triangle of sides a, b, c, find the lengths of the three medians.

369. Prove that the sum of the squares on the medians of a triangle equals three fourths the sum of the squares on the sides.

370. If two points in a diameter of a circle are equidistant from the center, the sum of the squares of their distances from a moving point on the circle is constant.

NOTE. The number of numerical exercises founded on the theorems of this book is practically unlimited. The pupil can familiarize himself with many of the different kinds by taking each one of the formulas and assuming values for all but one of the sects involved, and solving for that one. For example, if the sides of a triangle are a, b, c, and the median to c is m, then $a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2m^2$; assume values for (1) a, b, c, (2) a, b, m, (3) a, c, m, (4) b, c, m; thus using this formula in all possible ways.

In assuming lengths for lines in a triangle, care must be taken that the figure formed is possible; for instance, $a = 3$, $b = 7$, $c = 10$ is not possible. Why?

The best way to master a formula is to investigate each way it can be used, so that no question about it can be entirely unexpected.

SECTION III. CONSTRUCTIONS

249. CONST. I. (1) *To construct a polygon equivalent to a given polygon, but having one less side.*

(2) *To construct a triangle equivalent to a given polygon.*

Note that any two consecutive sides add to the surface only a triangular surface, which could equally well have been contained by one of these sides and the extension of one of the sides already used; in other words, it is a waste of a side to use *two extra sides to inclose an additional triangular surface.*

250. CONST. II. *To construct a rectangle equivalent to a given triangle.*

251. CONST. III. *To construct a square equivalent to the sum of two given squares.*

252. COR. 1. *To construct a square equivalent to the difference of two given squares.*

253. COR. 2. *To construct a square equivalent to the sum of any number of given squares.*

254. COR. 3. *To construct a square equivalent to any whole number of times a given square.*

371. Given the diagonals of two squares, construct the diagonal of the square equal to the sum of the given squares.

372. Construct a parallelogram equivalent to a given triangle, having its diagonals equal to two sides of the triangle.

373. Construct a rhombus equivalent to a given triangle, having one of its sides equal to a side of the triangle.

374. Construct a rectangle equivalent to any given polygon.

375. Construct a parallelogram equivalent to a given triangle, having one side equal to a side of the triangle, and having a given angle.

376. Construct a parallelogram equivalent to a given triangle, having its two sides equal to two of the sides of the triangle.

255. CONST. IV. *To construct a square equivalent to a given rectangle.*

Any of the formulas that give equivalences between squares and rectangles could be used to make this construction, for the given rectangle could represent the rectangle in the formula, and the squares could then be added or subtracted as stated in the formula. The formula for the difference of two squares [§ 230, (4)] is probably the best, although the difference of the squares on two sides of a triangle is almost, if not quite, as good. If the first way is used, what must the sides of the rectangle be called? Try to find as many constructions for this as possible, for they are excellent practice in the use of the equivalence formulas; there are at least seven methods that can be easily seen from the work done in this book.

377. Construct a rectangle equivalent to a given square, if one side of the rectangle is given.

378. Construct a rectangle equivalent to a given square, if the perimeter of the rectangle is given.

379. Construct a rectangle equivalent to a given square, if the diagonal of the rectangle is given.

256. SUMMARY OF THEOREMS AND COROLLARIES, BOOK III

(Numbers in parentheses refer to black-faced section numbers.)

I. PARALLELOGRAMS EQUIVALENT. = bases, bet. ‖s (231); = bases and = alt. (232); prod. bases and altitudes = (238).

II. TRIANGLES EQUIVALENT. = bases, bet. ‖s (231); = bases and = alt. (232); prod. bases and alt. = (238).

III. Comparison of Rectangles. Having = bases, proportional to alt. (233); having = alt., proportional to bases (234); proportional to the prod. of base and alt. (235).

IV. Areas. (1) rectangle = base × alt. (237).

(2) Parallelogram = base × alt. (238).

(3) Triangle = $\frac{1}{2}$ base × alt. (238); equilateral, = the square of half the side, times $\sqrt{3}$ (246).

(4) Trapezoid = $\frac{1}{2}$ sum of bases × alt. (239).

V. Equivalence Formulas involving Sects not Sides of a Triangle. (1) Square on sum of two sects equivalent to the sum of their squares plus twice their rectangle (230).

(2) Square on the difference of two sects equivalent to the sum of their squares less twice their rectangle (230).

(3) Square on twice a sect equivalent to four squares on the sect (230).

(4) Difference of the squares on two sects equivalent to the rectangle of their sum and their difference (230).

VI. Square on a Side of a Triangle. **General Theorem**: — The square on any side of any triangle is equivalent to the sum of the squares on the other sides plus twice the rectangle of one of those sides and the projection of the other on its line, that projection being considered positive or negative, according as it adds to, or takes away from, the side on which it is projected.

(1) In a right triangle, square on hypotenuse equivalent to the sum of the squares on the other two sides; square on a leg equivalent to the difference of the squares on the hypotenuse and the other leg (240, 241).

(2) Square on a side opposite an obtuse angle equivalent to the sum of the squares on the other sides plus twice the rectangle of one of those sides and the projection of the other on it (244).

(3) Square on a side opposite an acute angle equivalent to the sum of the squares on the other sides less twice the rectangle of one of those sides and the projection of the other on it (244).

VII. Squares on Two Sides of a Triangle. (1) Sum equivalent to twice the square on half the third side plus twice the square on the median to that side (247).

(2) Difference equivalent to twice the rectangle of the third side and the projection of its median on it (248).

CONSTRUCTIONS

VIII. TRIANGLE, equivalent to a polygon (249).
IX. RECTANGLE, equivalent to a triangle (250).
X. SQUARE, equivalent to
(1) sum of two squares (251).
(2) difference of two squares (252).
(3) sum of any number of squares (253).
(4) any number times a square (254).
(5) a rectangle (255).

257. ORAL AND REVIEW QUESTIONS

When is a square equivalent to the sum of two squares? to the difference? State the formula for the square on a side of a triangle opposite an obtuse angle; an acute angle. Does this apply to right triangles? Why? What is the method for finding the altitude of an equilateral triangle from the side? the area from the side? If the side is 24, find the altitude; the area. If the altitude is $19\sqrt{3}$, what is the side? If the area is $9\sqrt{3}$, what is the side? If the side is $8\sqrt{3}$, how far is it from the centroid to a vertex? How can the area of a triangle be most easily found from the sides? What kind of a triangle is one of sides 3, 4, 5? of sides 9, 7, 5? of sides 7, 8, 9? In what way can the sides of a trapezoid be used to find the area? Can the sides alone of a parallelogram be used to find the area? Why? What is needed? What angles can be found from the lengths of sides of triangles? in what kinds of triangles? What are the relative lengths of the sides of a triangle having angles of 90°, 60°, 30°? of 90°, 45°, 45°? If a leg of an isosceles triangle were known, and the vertex angle was 45°, how could one find the base? if the vertex angle were 30°? In constructing a triangle equivalent to a parallelogram, or *vice versa*, what relation between the surfaces should be kept in mind? Explain in a few words the general method used in reducing a polygon to a triangle. If it were necessary to draw a square equivalent to a given polygon, through what steps would it be necessary to go? a square equivalent to a triangle? to a parallelogram? to the sum of five squares? If a rectangle were to be constructed equivalent to a given square, and having a given side, what would it be best to call the side of the rectangle? Why? What three theorems give ratios between rectangles? Are all likely to be used for farther work, or have some served their purpose? Why was an extra figure constructed in one of them? What does area mean? When a theorem asks for the ratio of two like magnitudes for the first time, what must be done with those magnitudes? Then in terms of what is the ratio expressed? (NOTE. The commensurable case only is being considered.) Which formula do you regard as likely to be used the most frequently of all those in this book? Why? Explain two methods of finding the projection of one side of a triangle on another, when the sides are known. Explain two methods of finding the area of a triangle from the sides. Which is shorter? Which can be most readily used if formulas have been forgotten? What two methods of finding the median from the sides do you think of? Which is shorter? Which can be used if formulas are forgotten? What is the construction for finding the square on a side of an oblique triangle? Why? What is the formula for the area of a rectangle? a square? a parallelogram? a triangle? an equilateral triangle? a trapezoid?

GENERAL EXERCISES

380. If a square field and a rectangular field have the same area, which would require the longer fence?

381. Find the locus of a point such that the squares of its distances from two fixed points shall have a constant sum. (To prove a quantity constant, it is usually best to show that it *equals another quantity which is constant.*)

382. Find the locus of a point such that the squares of its distances from two fixed points shall have a constant difference.

383. If any point within a rectangle is joined to the vertices, the sum of the squares of the sects to two opposite vertices equals the sum of the squares of the sects to the other vertices.

384. The sum of the squares on the diagonals of any quadrilateral is equivalent to twice the sum of the squares on the sects joining the midpoints of the opposite sides.

385. Construct a rectangle equivalent to a given square, if the difference of the sides of the rectangle is given.

386. Find the three altitudes of the triangle of sides a, a, b.

387. The vertex angle of an isosceles triangle is 45°. Find the base, if the leg equals 10; if it equals a.

388. Find the area of a trapezoid of bases 12, 18, legs 4, 5.

389. Find the area of a trapezoid of bases a, c, legs b, d.

390. Find the side of an equilateral triangle of altitude 12.

391. If the altitude of an equilateral triangle is 10 in., what are the radii of the inscribed and circumscribed circles? if the side is 14 in.?

392. Compare the area of a square and a rhombus of the same sides, but with a 30° angle; a 60° angle; a 45° angle.

393. The sides of a parallelogram are 12 and 8. Find the area if an angle is 30°; if 60°; if 45°.

394. Construct a triangle having a given angle, a given side of a certain triangle, and equivalent to that triangle.

395. Construct a parallelogram equivalent to a given triangle, having its diagonals equal to two sides of the triangle.

396. Find the diagonals of the trapezoid in *388*.

BOOK IV. SIMILAR FIGURES. PROPORTIONS

SECTION I. RATIO AND PROPORTION

258. Terms of a Proportion. Ratio and proportion have already been defined (§ 202). The numerator of the first ratio and the denominator of the second ratio (the first and last terms) are called *extremes*, the other two terms are called *means*. The numerators of the two ratios are called *antecedents*, the denominators are called *consequents*. The last term of a proportion is called the *fourth proportional*.

259. Mean Proportion. If the means of a proportion are equal, the proportion is called a *mean proportion;* the mean is called the *mean proportional*, or simply *the mean*, and the last term is called the *third proportional*.

A *continued proportion* is a series of equal ratios in which any two successive ratios form a mean proportion.

260. Proportion Proofs. In proving the following proportion theorems, the letters a, b, c, d, may be used to represent like geometrical magnitudes expressed in terms of a common unit of measure. Then, while the ratio may be of two sects, or of two surfaces, that ratio takes a numerical form of expression; as, the ratio of two sects might be $\frac{5}{7}$.

These proofs must, of course, depend upon the equality axioms.

261. Composition; Division. Four quantities are said to be in proportion by *composition* when the antecedents become the sums of the terms of the ratios; as,

$$\frac{a+b}{b}=\frac{c+d}{d}.$$

Four quantities are said to be in proportion by *division* when the antecedents become the differences of the terms of the ratios; as,

$$\frac{a-b}{b}=\frac{c-d}{d}.$$

If the ratios of the sums to the differences are used, the quantities are said to be in proportion by *composition and division;* as,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

262. Equimultiples. *Equimultiples* of two quantities are the results obtained by multiplying those quantities by the same number.

NOTE. In proving the theorems in Ratio and Proportion, use the fractional form for the ratios, and use the proportion as an equation; as, $\frac{a}{b}=\frac{c}{d}$. In this way the equality axioms can be used more easily.

THEOREMS

263. Theorem I. *If four quantities are in proportion, the product of their means equals the product of their extremes.*

264. COR. 1. *In a mean proportion, the square of the mean equals the product of the extremes.*

265. COR. 2. *The value of any term of a proportion can be expressed in the other terms of the proportion.*

266. Theorem II. *If the product of two quantities equals the product of two other quantities, either pair can be made the means, and the other pair the extremes, of a proportion.*

267. Cor. 1. *If four quantities are in proportion, they are in proportion in any way in which the means of the given proportion are either both means, or both extremes, in the new proportion.*

397. Any two sides of a triangle are inversely proportional to the altitudes drawn to them. (*Inversely* proportional means that one of the ratios is inverted.) Use area formula.

268. Theorem III. *Four quantities which are in proportion are in proportion by composition.*

269. Theorem IV. *Four quantities which are in proportion are in proportion by division.*

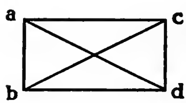
270. Theorem V. *Four quantities which are in proportion are in proportion by composition and division.*

271. Theorem VI. *If four quantities are in proportion, equimultiples of the antecedents are in proportion to equimultiples of the consequents.*

272. Theorem VII. *If four quantities are in proportion, like powers of those quantities are in proportion.*

273. Theorem VIII. *In a series of equal ratios, the ratio of the sum of the antecedents to the sum of the consequents equals any of the given ratios.*

274. **SUMMARY**



If a proportion is written in fractional form, and the four terms are considered as forming the vertices of a rectangle, its terms will be in proportion in any order in which the pairs are taken along opposite sides of the rectangle, in the same direction;

that is, both to the right, both down, etc. For example, starting from d and going toward the left, $\frac{d}{b}=\frac{c}{a}$.

Notice that the proportion never goes along a diagonal, as from a to d; this can be kept in mind because the diagonals form a multiplication sign, and diagonal terms can be multiplied but not divided.

The same method applies to composition and to division, the same operations being applied to opposite sides to form the new antecedents, and to form the new consequents. For example, starting from b and going up, $\frac{b+a}{b-a}=\frac{d+c}{d-c}$.

This method can be extended to apply to equimultiples, to powers, and to composition and division forms involving equimultiples and powers, and in this way it serves as a test of the correctness of proportion forms.

This is not a proof, but simply a test for correctness, which also acts as a help to the memory by combining all the most important proportion forms in one rule.

SECTION II. PROPORTIONAL SECTS

275. Pencil of Lines. Lines that are concurrent are spoken of as a *pencil of lines.* In the same way a number of lines that are all parallel are spoken of as a *pencil of parallels.*

276. Theorem I. *If a line is cut by a pencil of parallels, its sects are proportional to the sects of any other line cut by the same pencil of parallels,* including as a special case,

A line parallel to the base of a triangle cuts the sides, or the sides extended, so that the sects are proportional. (Com. case.)

277. Cor. 1. *A line parallel to the base of a triangle has the same ratio to the base as the lengths it cuts off on the other sides (from their common vertex) have to the whole sides.*

278. Cor. 2. *If parallel lines are cut by a pencil of lines, the sects cut off on the parallels are proportional.*

398. In the quadrilateral $ABCD$, having angle B and angle D right angles, PE and PF are drawn from P in AC perpendicular to BC and DA, respectively. Prove that $BE:EC = AF:FD$.

399. If BC, of triangle ABC, is extended to X, and AY is cut off on AB equal to CX, then XY is cut by CA in the ratio $AB:BC$.

400. The diagonals of a trapezoid cut each other proportionally, and their sects are proportional to the bases.

401. If a line cuts the sides of a triangle, extended if necessary, the product of three non-consecutive sects equals the product of the other three sects.

279. Points Cutting a Sect; Harmonic Division. A point on a sect, or on the sect extended, is said to cut the sect in the ratio of its distances from the ends of the sect; as, if P is on AB, or on AB extended, the ratio in which it cuts AB is $PA : PB$. If the point is three fourths as far from A as from B, it cuts AB in the ratio $3 : 4$, and this is the same whether P is in the sect itself or not.

If two points cut the same sect in the same ratio, one internally, the other externally, they are said to cut the sect *harmonically*. The equal ratios must, of course, be taken from corresponding ends of the sect; as, if P and Q cut AB harmonically, $PA : PB = QA : QB$.

Notice that a sect cannot be cut externally in the ratio 1, for if P is in the extension of AB, PA cannot equal PB.

280. Theorem II. *A sect can be cut in the same ratio internally by but one point, and externally by but one point.*

402. The interior common tangents of two circles (those between the circles) meet the center line at the same point.

403. The exterior common tangents of two circles meet the center line at the same point.

404. The interior and exterior common tangents to two circles cut the center sect harmonically.

405. A line through the ends of two parallel radii of two circles meets the center line at the same point as the common tangents.

NOTE. The points where the common tangents meet the center line are called the *inverse, and direct centers of similitude*.

406. If two points cut a sect harmonically, they include a second sect, which is cut harmonically by the ends of the first sect.

281. Theorem III. *A line that cuts two sides of a triangle proportionally is parallel to the third side.*

407. If a sect joins the one third points of two sides of a triangle (taken from their common vertex), what part of the third side is it?

SECTION III. SIMILAR FIGURES

282. Similar Figures. Polygons are said to be *similar* if their corresponding angles are equal and their corresponding sides are proportional. See also Appendix, § 349.

There are now three things which can be proved about polygons: that they are congruent, equivalent, or similar. Equivalent means of the same size (as regards surface), similar means of the same shape, while congruent includes both size and shape. Notice that the sign for congruent is composed of the equivalent sign and the similar sign. These facts are not definitions of the words, but serve to show the distinction in meaning in a somewhat different light.

***283.** *Polygons similar to the same polygon are similar to each other.*

***284.** *Perimeters of similar polygons are proportional to any pair of corresponding sides.*

***285.** *Regular polygons of the same number of sides are similar.*

408. If two similar polygons are placed with a pair of corresponding sides parallel (the polygons lying on the same sides of those lines), the lines through the corresponding pairs of vertices will form a pencil, which is cut proportionally by the vertices of the polygon.

409. If two polygons lie in a pencil of lines, and their vertices cut the lines proportionally, the polygons are similar.

286. Theorem IV. *Two triangles are similar if two angles of one are equal to two angles of the other.*

410. All lines through the point of tangency of two circles are cut proportionally by the circles.

411. If AB is a diameter of a circle, CD a chord perpendicular to AB, then any chord AY cutting CD at X has the product $AX \times AY$ constant.

412. The product of two sides of a triangle equals the product of the altitude to the third side by the diameter of the circumscribed circle.

413. The product of two sides of a triangle equals the product of the bisector of the included angle by the sect of that bisecting line from the vertex of the angle to the circumscribed circle.

287. Theorem V. *Two triangles are similar if two sides of one are proportional to two sides of the other, and the included angles are equal.*

414. In any triangle, the orthocenter, the centroid, and the circumcenter, lie in a straight line, and the distance between the first two is double the distance between the second two.

288. Theorem VI. *Two triangles are similar if the sides of one are proportional to the sides of the other.*

If one had the included angle equal to that of the other, the triangles would be similar; cut it off equal, and show that the triangle obtained is the same triangle as that given.

289. Similar Triangles. Similar triangles are obtained much as congruent triangles were obtained, namely, by three parts. The angles are given equal, but the sides are given proportional instead of equal. Any three parts will do, except two sides proportional and a pair of angles, not included, equal. The following table will serve to show the relation between congruence and similarity.

Given Parts	If Sides are Equal	If Sides are Proportional
3 sides:	figures congruent.	figures similar.
2 sides, angle:		
included,	figures congruent;	figures similar.
not included,	figures congruent if angle is right or obtuse; or if acute, with the greater side opposite. Otherwise, other angles not included are supplemental.	same conditions as for congruent.
1 side, 2 angles:	figures congruent.	figures similar; side not needed as two sects cannot be proportional; same as 3 angles.
3 angles:	figures not necessarily congruent.	figures similar.

Similar figures, and especially similar triangles, have many practical applications, such as in finding the height of trees and buildings, and the distances between objects. The principle involved is that if two triangles are similar, any one side, of two pairs of corresponding sides, can be found if the other three are known. Some of the following exercises illustrate this method.

415. How tall is the tree that casts a shadow 50 ft. long at the same time that a pole 6 ft. long casts a shadow 8 ft. long?

416. A building casts a shadow 64 ft. long. A projection on one corner of the building that is found to be 8 ft. from the ground casts a shadow 9 ft. long. How high is the building?

417. It is necessary to find the distance from B to an inaccessible point X. A line BA perpendicular to the sighted line BX is laid off 146 ft. long, and at K, 60 ft. from A on AB, a perpendicular to AB is erected, meeting the sighted line AX at L. If KL is found to be 30 ft. long, how long is BX?

418. Two triangles are similar if the sides of one are parallel to the sides of the other.

419. Two triangles are similar if the sides of one are perpendicular to the sides of the other.

420. If two chords of a circle cut each other, the four sects are proportional.

421. If two secants of a circle intersect, the four sects from the vertex to the circle are proportional.

NOTE. There are now two principal methods by which to find four sects proportional. What are they?

290. Theorem VII. *The areas of triangles, or of parallelograms, having an angle of one equal to an angle of the other, have the same ratio as the product of the sides including that angle.*

291. COR. 1. *The areas of similar triangles have the same ratio as the squares of their corresponding sides.*

292. COR. 2. *If two triangles that have an angle of one equal to an angle of the other are equivalent, the product of the sides including the angle in one equals the product of the sides including the angle in the other, and conversely.*

422. If BC, of triangle ABC, is 8, and CA is 6, how long must CY be, so that a line from Y, on BC, to X, the two thirds point of CA, will cut the triangle into equivalent parts.

423. A line from the midpoint of a side of a triangle must go to what point on a second side to form an equivalent triangle?

424. In a right triangle of legs 3 and 4 ft., the hypotenuse is extended 10 ft. How long must a leg be extended at the same vertex so that the line joining the extremities of the extensions will form a triangle double the given triangle? (Two cases.)

425. If similar triangles are drawn upon the sides of a right triangle as corresponding sides, the triangle on the hypotenuse equals the sum of the other triangles.

293. Theorem VIII. *If two polygons are similar, they can be divided into the same number of triangles, similar each to each, and similarly placed.*

294. Theorem IX. *If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, the polygons are similar.*

295. Theorem X. *The areas of similar polygons have the same ratio as the squares of any pair of corresponding sides.*

426. If similar polygons are constructed on the sides of a right triangle as corresponding sides, the polygon on the hypotenuse equals the sum of the other two polygons.

427. If each side of one polygon is double the corresponding side of a second similar polygon, what relation have the areas?

428. If the area of one polygon is 36 times the area of a similar polygon, what relation have the sides?

NOTE. *Area ratios in similar figures are always the squares of line ratios; line ratios are the square roots of area ratios.*

296. Theorem XI. *In a right triangle, the altitude to the hypotenuse divides the triangle into two triangles similar to each other, and to the whole triangle.*

297. COR. 1. (1) *The altitude to the hypotenuse is the mean proportional between the sects of the hypotenuse.*

(2) *Either leg of the triangle is the mean proportional between the hypotenuse and its own projection on the hypotenuse.*

429. In a right triangle of legs 5 and 12, find the projections of 5 and of 12, on the hypotenuse; find the altitude to the hypotenuse.

430. In a right triangle of legs a, b, hypotenuse c, find the projections of a and of b on c; find the altitude to c.

NOTE. The results of exercise 430 are formulas that hold for all right triangles.

431. Tangents from a point to a circle of radius 6 are of length 8. Find the chord of contact.

432. In a right triangle, the sects of the hypotenuse made by the altitude are 4 and 5. Find the other sides and the altitude.

433. The squares of two chords drawn from a point on a circle have the same ratio as their projections on the diameter from that point.

434. The half chord perpendicular to a diameter is the mean between the sects of the diameter.

298. Theorem XII. *A line that bisects an angle, interior or exterior, of a triangle, divides the opposite side, internally or externally, into sects proportional to the other sides of the triangle.* Note that the two lines divide harmonically.

435. If a line drawn from the vertex of an angle of a triangle divides the opposite side, internally or externally, in the ratio of the other two sides, the line bisects the angle, interior or exterior, from whose vertex it is drawn.

436. In a triangle of sides 6, 7, 8, find the sects of 7 made by the bisector of the opposite angle; of the exterior angle at the opposite vertex.

437. In a triangle of sides a, b, c, find the sects of c made by the bisector of the opposite angle; of the exterior angle at the opposite vertex.

NOTE. The results of *437* are formulas that hold for all triangles.

438. If two lines from the vertex of the right angle of a right triangle make equal angles with one of the legs, they cut the hypotenuse harmonically.

299. Theorem XIII. *If a pencil of lines cuts a circumference, the lines are cut proportionally, so that the product of the two sects from the vertex to the circumference on one line is the same as that product on any other line.*

300. Cor. 1. *A tangent from the vertex of a pencil to a circumference is the mean proportional between the two sects, from the vertex to the circumference, of any other line of the pencil that is cut by the circumference.*

439. If two points are taken on each line of a pencil, so that the product of the two sects from the vertex is the same for all the lines of the pencil, the two points being on opposite sides of the vertex, the four points on any two lines are concyclic.

440. In the figure of *439*, if the two points on each line are on the same side of the vertex, the four points on any two lines are concyclic.

441. Find the locus of the point P on a secant to a given circle which cuts the circle in changing points, A and B, so that $PA \times PB$ is constant.

442. If two intersecting lines are cut, one by one point, the other by two points on the same side of the vertex, so that the sect cut off by the one point is the mean proportional between the sects from the vertex on the other line, a circle through the three points would be tangent to the line cut by the one point.

443. Construct a circle through two given points tangent to a given line.

444. The product of two sides of a triangle equals the square of the bisector of the included angle, plus the product of the sects into which the bisector divides the opposite side.

Suggestion. Inscribe the triangle.

445. If three circles intersect, their common chords are concurrent.

446. If two circles intersect, tangents from any point in their common chord extended, to the two circles, are equal.

SECTION IV. CONSTRUCTIONS

301. Const. I. *To divide a given sect internally, and externally, in a given ratio (or, harmonically).*

Note. A given ratio is always represented by two given sects, the given ratio being that of those sects.

302. Cor. 1. *To divide a given sect into parts proportional to any number of given sects.*

447. Given a sect, and one point of harmonic division, find the other.

448. Given a sect, cut it into parts having the ratio $1:2:5$.

303. Const. II. *To find the fourth proportional to three given sects.*

304. Cor. 1. *To find the third proportional to three given sects.*

449. Draw a line through a given point so that it will cut off sects having a given ratio on the arms of a given angle.

450. Draw a line from a given point to a given line, so that it will have a given ratio to the perpendicular from that point.

451. Draw a line through a given point so that it will be cut in a given ratio by the arms of the angle.

452. Construct two sects, given their sum and their ratio.

453. Construct two sects, given their difference and their ratio.

305. Const. III. *To find the mean proportional between two given sects.* Find three methods.

This construction is the foundation of all square root questions in the Geometry. Since the square of the mean equals the product of the extremes, it follows that the mean equals the square root of the product of the extremes. It is, therefore, possible to construct the square root of any required number by making the extremes of such length that their product equals the number of which the square root is asked. This is used also in constructing figures whose areas have a certain ratio, for if the figures are similar, the line ratio is the *square root* of the given area ratio, and so can be found by the mean proportional.

454. Construct a sect $\sqrt{2}$ in. long, given the sect 1 in. long.

455. Explain how the sect $\sqrt{n}$ in. long could be constructed for any value of n, given a sect of 1 in.

456. Find another way to construct the square root of a number.

306. Mean and Extreme Ratio. If a sect is divided so that the longer part is the mean between the whole sect and the shorter part, the sect is said to be divided in *mean and extreme ratio*. If AB is divided by P so that $\frac{AB}{AP}=\frac{AP}{PB}$, then P divides AB in mean and extreme ratio. There is an external mean and extreme division as well as an internal; the words "mean" and "extreme" do not refer to the interior and exterior cases, but to the position in which the parts occur in the proportion.

It should be noticed that, as the proportion stands, there are two unknown sects used; by a proper transforming of the proportion the parts can be combined (in different ways for the two cases) so that the given sect AB will appear more often, thus displacing the unknowns. The new form, in which the unknowns occur as seldom as possible, is the one with which to work in attempting the construction; it will be found very easy if attacked logically.

307. CONST. IV. *To divide a given sect in mean and extreme ratio.*

457. If a line 10 in. long is divided in mean and extreme ratio, how long is the mean sect?

458. If the bisector of a base angle of an isosceles triangle cuts off the longer sect of the opposite side equal to the shorter of the sides including the bisected angle, that side is cut in mean and extreme ratio.

308. CONST. V. *To construct a polygon similar to a given polygon, on a given sect as a side corresponding to a certain side of the given polygon.*

459. Construct a polygon similar to a given polygon and having twice as great an area; n times as great an area (n a whole number).

460. Construct a polygon similar to a given polygon and having an area one half as great; three fifths as great; $\frac{n}{m}$ times as great.

461. Given two similar polygons, construct a polygon similar to them and equal to their sum.

462. Given two similar polygons, construct a polygon similar to them and equal to their difference.

309. CONST. VI. (1) *To draw a rectangle having a given ratio to a given square.* (2) *To draw a square having a given ratio to a given square.*

Notice that the construction "To draw a square equivalent to a given rectangle" uses a method of construction which is the same as drawing a mean proportional. Why?

463. Construct a square three fourths as large as a given square.

464. Construct a square five times as large as a given square by the method of § 309.

310. **SUMMARY OF THEOREMS AND COROLLARIES, BOOK IV**

(Numbers in parentheses refer to black-faced section numbers.)

I. SECTS PROPORTIONAL. If cut off on transversals by ‖s (276); cut off on ‖s by a pencil (278); sides of △ cut by a ‖ to third side (276);

line parallel to side of △ has the same ratio to it as the length cut off on another side has to that side (277); sides of ∼ polygons (282); perimeters of ∼ polygons proportional to sides (284); in a right triangle the altitude is the mean, either leg is a mean (297); line bisecting an angle of a △ divides the opposite side (298); pencil cutting a circle, the product of the sects is constant (299); the tangent is the mean (300).

II. SECT CUT. By but one internal point in a given ratio, by but one external point in a given ratio (280).

III. LINES PARALLEL. Line cutting two sides of △ proportionally (281).

IV. FIGURES ARE SIMILAR. ∼ to the same polygon (283); regular, same number of sides (285); ⊿s, if they have 2 ∠s = (286), 2 sides proportional, incl. ∠s = (287), 3 sides proportional (288); in rt. △ altitude forms ⊿s ∼ each other and to the whole (296); ∼ polygons can be divided into ∼ ⊿s (293); polygons composed of ∼ ⊿s are ∼ (294).

V. AREA RATIOS. ⊿s having an = ∠ (290); ▱s having an = ∠ (290); ∼ ⊿s proportional to squares of sides (291); ∼ polygons proportional to squares of sides (295); equivalent when an angle =, product including sides the same (292).

CONSTRUCTIONS

VI. DIVIDE A SECT. Internally and ext. in a given ratio (301); into parts proportional to given sects (302); in mean and extreme ratio (307).

VII. FIND A PROPORTIONAL. Fourth to three given sects (303); third to two given sects (304); mean between two given sects (305).

VIII. CONSTRUCT A POLYGON. Similar to a given polygon (308).

IX. CONSTRUCT A SQUARE. Having a given ratio to a given square (309).

311. ORAL AND REVIEW QUESTIONS

What are the two principal ways of getting sects proportional? Upon what do area ratios always depend? How many parts are needed to prove triangles similar? Will any three parts do? If one side of a polygon is double the corresponding side of a similar polygon, what can be told about the areas? If the area is double, what can be told about the side? If each side of a triangle including one of the angles is doubled, what does it do to the area of the triangle?

if one side is made three halves as long, the other two thirds as long? How could a triangle be made isosceles without changing the vertex angle or the area? Why are polygons similar to the same polygon similar to each other? In proving polygons similar, what two things must be considered? Why are regular polygons of the same number of sides mutually equiangular? Why are their sides proportional? What is used when the ratio of a sum of several magnitudes to a second sum is needed? What changes can be made in a proportion without making it untrue? If the sides of a right triangle are 3, 4, 5, how long is the projection of 3 on 5? of 4 on 5? How long is the altitude to 5? How long are the sects of 5 made by the bisector of the opposite angle? Tell three ways to construct a mean proportional. What is the foundation of the constructions which make four sects proportional? Name three such constructions. What constructions in this book have special cases which have been done before? What new way of finding that a line is parallel to another line occurs in this book? What special case of it has been taken before? What other theorem is the general case corresponding to a special case done before? Define mean and extreme ratio, cutting harmonically, similar polygons, fourth proportional, composition, division. In what form is the mean and extreme ratio proportion put before trying to find the internal point of division? the external? What is meant by a point cutting a sect in a given ratio internally? externally? In what ratio is it not possible to cut a sect? What is the fundamental way of proving four sects proportional? What two cases of it are there? If a line were drawn through the two thirds points of the legs of a trapezoid, would it be parallel to the bases? Why? What new way of proving triangles equivalent has been found? Give a numerical example illustrating it. How can a line of length $\sqrt{2}$ be constructed? the length $\sqrt{n}$? Is the length exact or approximate (in so far as the instruments are accurate)?

GENERAL EXERCISES

465. If two lines through the vertex of a triangle divide the opposite side harmonically, and are at right angles to each other, they bisect the angles at that vertex.

466. Find the locus of a point whose distances from two given points are in a given ratio.

467. Find the locus of a point such that the angle between the tangents from the point to one given circle equals the angle between the tangents from the point to a second given circle.

468. Construct a square inscribed in a semicircle.

469. A line is drawn from a vertex of a parallelogram, cutting the other diagonal, and the other two sides, one extended. Prove that the sect from the vertex to the diagonal is the mean between the sects from the diagonal to the sides.

470. If the center line of two circles that do not meet intersects the exterior common tangents at P, and the circles at A, B, C, D, and a secant from P meets the circles at X, Y, Z, W, then $PX \times PW = PY \times PZ = PB \times PC$.

471. Find the locus of the vertex of a triangle, given the base, and the ratio of the other sides.

472. Find the locus of a point whose distances from two intersecting lines have a given ratio.

473. Find the locus of a point whose distances from the sides of a triangle are in a given ratio.

474. Cut off equal parts from two given sects, leaving them in a given ratio.

475. Inscribe a triangle similar to a given triangle in a given circle.

476. Circumscribe a triangle similar to a given triangle about a given circle.

477. From a given point on a side of a triangle, draw a line dividing the triangle into equivalent parts.

478. Construct an equilateral triangle equivalent to a given triangle.

479. Construct a sect, given the greater sect obtained by dividing it in mean and extreme ratio.

480. Inscribe a square in a given triangle.

481. Inscribe a rectangle similar to a given rectangle in a given semicircle.

482. Inscribe a rectangle similar to a given rectangle in a given triangle.

483. In a triangle of sides a, b, c, find the radius of the circumscribed circle.

484. Draw a square equivalent to a given rhombus.

485. Describe a circle through a given point, tangent to two given lines.

486. Construct a triangle, given the three altitudes. Note that the ratio of the altitudes can be used to find the ratio of the sides.

487. Construct an isosceles triangle, given the vertex angle and the sum of the base and its altitude.

488. Construct a triangle, given the base angles, and the difference between the base and its altitude.

NOTE. In many triangle and other constructions, a similar figure can be used to advantage; as here, a triangle similar to the required triangle should be constructed first.

489. Construct a triangle, given the base, a base angle, and the ratio of the other including side to the radius of the circumscribed circle.

490. Construct a triangle, given the three medians.

NOTE. Many other triangle constructions can be solved by the use of proportions and of similar figures. It is good practice for students to try to invent new combinations of parts of triangles from which the triangles can be constructed. See also note at end of Bk. II.

BOOK V. REGULAR POLYGONS AND CIRCLES

SECTION I. THEOREMS

312. Theorem I. *In any regular polygon there is a point that is equidistant from the vertices and equidistant from the sides.*

Prove that the bisectors of all the angles meet in a point, and use locus.

313. Center, Radius, Apothem. That point in a regular polygon which is equidistant from the vertices, and also equidistant from the sides, is called the *center* of the polygon; the line from the center to a vertex is called the *radius* of the polygon, and a perpendicular from the center to a side is called the *apothem* of the polygon.

It is evident that the radius of the polygon is also the radius of the circumscribed circle, and that the apothem of the polygon is the radius of the inscribed circle.

314. Cor. 1. *The area of a regular polygon equals one half the product of the perimeter by the apothem.*

315. Cor. 2. *The perimeters of regular polygons of the same number of sides are proportional to their sides, apothems, or radii.*

316. Cor. 3. *The areas of regular polygons of the same number of sides are proportional to the squares of their sides, apothems, or radii.*

Note again that the area ratio is the *square* of the line ratio.

491. If from a point within a polygon of n sides perpendiculars are drawn to all the sides, the sum of those perpendiculars is n times the apothem.

492. Find the area of a regular hexagon of side 10; of side a.

493. If the apothem of one regular polygon is 12, that of another of the same number of sides is 15, and the area of the first is 477.16, find the area of the second.

317. Theorem II. *An equilateral polygon inscribed in a circle is regular; an equiangular polygon circumscribed about a circle is regular.*

318. Variables and Limits. A *constant* quantity is one that keeps the same value throughout the investigation in question. A quantity may be constant in one discussion, but not in another.

A *variable* quantity is one that takes different successive values during an investigation.

The *limit* of a variable is that constant to which the variable can approach so near that the difference is less than any possible fixed quantity, but which the variable cannot equal.

If the sum of the numbers 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., is taken, that sum will never equal 2, no matter how large a number of terms is added. However, it is not possible to name a number less than 2, such that the sum cannot become greater than that number; that is, be nearer to 2 (its limit) than any fixed number.

It is evident from the definition that when a variable approaches its limit, the difference between the limit and the variable approaches the limit zero; and, conversely, that when the difference between a constant and a variable approaches zero as a limit, the variable must be approaching the constant as its limit.

319. Limit Theorems. (Given without proof; see Appendix, § 347.)

(1) *If two variables approaching limits are equal for all values, their limits are equal.*

(2) *If a variable is approaching a limit, that variable multiplied by, or divided by, any constant will approach its limit multiplied by, or divided by, that constant.*

(3) *If two variables are proportional to two constants, their limits are proportional to the same constants.*

320. Theorem III. *If the number of sides of a regular polygon inscribed in a circle is increased indefinitely, the apothem of the polygon will approach the radius of the circle as a limit.*

Show that $r - a < \frac{s}{2}$, where r, a, s, stand for radius, apothem, and side; then show that $s \doteq 0$, when the number of sides is increased indefinitely.

321. Circumference Axiom. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of sides is increased indefinitely.*

322. Theorem IV. *The area of a circle is the limit which the areas of regular inscribed and circumscribed polygons approach when the number of sides is increased indefinitely.*

323. Theorem V. *The ratio of the circumference to the diameter is the same for all circles* (or, *circumferences are proportional to their diameters*).

Regular inscribed polygons have perimeters proportional to the diameters; apply limit Th. III.

324. Value of the Ratio of Circumference to Diameter. The ratio of circumference to diameter is represented by the Greek letter π (called *pi*). This letter is the initial letter of the Greek word for circumference. The value of π can be found numerically by Geometry, and the method employed is shown in the Appendix (§ 350); the value commonly used in calculations is 3.14159, or for less accurate results, 3.1416, or even $3\frac{1}{7}$. The last value is sufficiently accurate for many of the numerical exercises of the Geometry, but the student should become accustomed to the use of the more accurate values also.

The value of π cannot be expressed exactly in the decimal system; that is, it is an incommensurable number. It has, however, been calculated to over 700 decimal places, and there is no limit to the accuracy with which a calculation can be carried out, if it is considered worth while.

325. COR. I. *The circumference of any circle equals* 2π *times its radius.*

494. If the radius of one circle is double the radius of a second circle, and the circumference of the second circle is 30 ft. long, how long is the circumference of the first circle?

495. What is the width of the ring between two concentric circles whose circumferences are 100 and 200 ft.

496. If one third of the circumference of one circle equals one fourth the circumference of a second circle, how do the radii compare?

497. Find the circumference of a circle of radius 10 in.

498. Find the radius of a circle of circumference 22 ft.

326. Theorem VI. *The area of a circle is one half the product of the circumference by the radius.*

Use a circumscribed regular polygon.

327. Cor. 1. *The area of a circle equals* π *times the square of the radius.*

328. Cor. 2. *The areas of two circles are proportional to the squares of their radii.*

499. Find the area of a circle of radius 10.

500. Find the radius of a circle of area 49.

501. Find the circumference of a circle of area 3.14159.

502. If the radius of one circle is four times the radius of a second circle, the area of the first is how many times the area of the second?

503. If the circumference of one circle is twice the circumference of a second, how do the areas compare?

504. What is the area of the ring between concentric circles of circumferences 100 and 200 ft.?

SECTION II. CONSTRUCTIONS

329. Const. I. (1) *To inscribe a circle in a given regular polygon.*

(2) *To circumscribe a circle about a given regular polygon.*

330. Const. II. (1) *Given a regular inscribed polygon, to draw the regular circumscribed polygon of the same number of sides.*

(2) *Given a regular circumscribed polygon, to draw the regular inscribed polygon of the same number of sides.*

331. Const. III. (1) *Given a regular inscribed polygon, to draw the regular inscribed polygon of double the number of sides.*

(2) *Given a regular circumscribed polygon, to draw the regular circumscribed polygon of double the number of sides.*

332. Const. IV. *To inscribe a square in a given circle.*

333. Const. V. *To inscribe a regular hexagon in a given circle.*

334. Const. VI. *To inscribe a regular decagon in a given circle.*

505. Construct a regular pentagon inscribed in a given circle.

506. Construct a regular 12-sided figure inscribed in a given circle.

507. Circumscribe about a given circle a polygon of 4, 5, 6, 8, 10, sides.

335. Regular Polygons which can be Constructed. It is clear from the constructions in this book that polygons of 3, 4, 5, sides, or those numbers multiplied by any power of 2, are possible. There are certain other regular polygons which can be constructed with compass and ruler, such as the polygon of 17 sides, but the ones mentioned, and possibly the one of 15 sides, are the only ones of importance to Elementary Geometry.

508. Construct a regular polygon of 15 sides.

336. SUMMARY OF THEOREMS AND COROLLARIES. BOOK V

(Numbers in parentheses refer to black-faced section numbers.)

I. Regular Polygon. Has a center (312); area $= \frac{1}{2}$ ap. $\times$ per. (314); perimeters prop. to sides, apothems, radii (315); areas prop. to squares of sides, apothems, radii (316); equilateral insc. polygon regular, equiangular circumscribed polygon regular (317).

II. Limit Applications. Apothem of regular inscribed polygon approaches radius (320); circumference axiom (321); area regular insc. or circum. polygon approaches the area of the circle (322).

III. Circles. Ratio of circumference to diameter (π) the same for all circles (323); circum. $= 2\,\pi r$ (325); area $= \frac{1}{2}\,r \times$ circum. (326); area $= \pi r^2$ (327); areas prop. to squares of radii (328).

CONSTRUCTIONS

IV. To draw a Circle. Inscribed in a given regular polygon (329); circumscribed about a given regular polygon (329).

V. To draw a Regular Polygon. Circumscribed, of same number of sides as a given inscribed, or inscribed, of the same number of sides as a given circumscribed (330); of double the number of sides of a given polygon, inscribed or circumscribed (331); of 4 sides inscribed (332); of 6 sides inscribed (333); of 10 sides inscribed (334).

337. **ORAL AND REVIEW QUESTIONS**

How is the center of a regular polygon found? Tell a second way. To what are the perimeters of regular polygons proportional? the areas? the circumferences of circles? the areas of circles? Give a formula for the circumference of a circle, two formulas for the area of a circle. To what is the side of a regular inscribed hexagon equal? the side of a regular inscribed decagon? the side of a circumscribed square? of an inscribed square? of a circumscribed equilateral triangle? of an inscribed equilateral triangle? What must be known to prove an inscribed polygon regular? a circumscribed polygon? Explain the inscribed case. What is the construction upon which the regular decagon depends? How can a regular pentagon be drawn? Tell which of the following regular polygons can be constructed with compass and ruler, and explain; 7, 8, 9, 12, 14, 16 sides. What values of π are most commonly used? What is the circumference of a circle of radius 10? the area? How can you find the radius from the circumference? the radius from the area? the circumference from the area? the area from the circumference? How can circumferences be added? subtracted? multiplied? How can the areas of circles be added? subtracted? multiplied? divided? Explain how to draw a circle having its circumference double that of a given circle; its area double that of the given circle.

GENERAL EXERCISES

509. The area of a regular dodecagon (12) equals three squares on its radius.

510. What is the radius of a circle, if the area of the regular inscribed hexagon is $6\sqrt{3}$?

511. The area of the ring between two concentric circles equals that of a circle whose diameter is that chord of the larger which is tangent to the smaller.

512. If two chords of a circle are perpendicular to each other, the sum of the circles on the sects as diameters equals the original circle.

513. On the sides of a square of side a, as diameters, circles are drawn. Find the area of the parts into which the square is divided.

514. On the sides of an equilateral triangle of side a as diameters, circles are drawn. Find the areas of the parts of the figure formed.

515. With the vertices of an equilateral triangle of side a as centers, and a radius equal to half the side, circles are drawn. Find the area of the entire figure, and of the figure inside the triangle bounded by the arcs.

516. Construct a regular octagon on a given sect as side.

517. Construct a regular hexagon on a given side.

518. Construct a regular decagon on a given side.

519. Construct a regular pentagon on a given side.

520. Construct a regular hexagon, given the shorter diagonal.

521. Construct a circle equal to the sum of two given circles.

522. Construct a circle equal to the difference of two given circles.

523. Construct a circle equal to the sum of any number of given circles.

524. Construct a circle equal to any number of times a given circle.

525. Construct a circumference equal to the sum of two given circumferences.

526. Construct a circumference equal to the difference of two given circumferences.

527. Construct a circumference equal to the sum of any number of given circumferences.

528. Construct a circle whose area has any given ratio to the area of a given circle.

529. Construct a circle whose area is one half the area of a given circle.

530. Divide a given circumference into parts having the ratio 3 : 7, by a line through a given point.

531. Divide the surface of a circle into equal parts by a concentric circle.

532. Divide the surface of a circle into any number of equal parts by concentric circles.

533. Given the radius of a circle, find the perimeters of regular inscribed and circumscribed polygons of 3, 4, 5, 6, 8, 10 sides.

534. Cut off two thirds of a circle by a line through a given point.

GENERAL

THE FORMULAS OF GEOMETRY

338. I. In an n-sided Polygon.

(1) The sum of the interior angles equals $(n - 2)$ st. ∠s.

(2) The sum of the exterior angles equals 2 st. ∠s.

II. Angles Formed by Lines Meeting a Circumference.

(1) Vertex on the circumference, measured by half the arc; includes inscribed angles (three cases), tangent and chord angles.

(2) Vertex inside the circle, measured by half the sum of the arcs.

(3) Vertex outside the circle, measured by half the difference of the arcs; includes angles between two secants, two tangents, tangent and secant; angle between two tangents supplemental to central angle between radii to points of tangency.

III. Area.

(1) Rectangle, equals base times altitude.

(2) Parallelogram, equals base times altitude.

(3) Triangle, equals half base times altitude.

(4) Triangle, given the sides, equals the square root of $s(s-a)(s-b)(s-c)$, where s is half the sum of the sides.

(5) Equilateral triangle, equals the square of half the side times the square root of three.

(6) Trapezoid, equals half the sum of the bases times the altitude.

(7) Regular polygon, equals half the perimeter times the apothem.

(8) Circle, equals π times the square of the radius.

IV. EQUIVALENCE FORMULAS BASED ON THE AXIOM OF THE WHOLE.

(1) The square on the sum of two sects is equivalent to the sum of their squares plus twice their rectangle.

(2) The square on the difference of two sects is equivalent to the sum of their squares less twice their triangle.

(3) The square on twice a sect is equivalent to four squares on the sect.

(4) the difference of the squares on two sects is equivalent to the rectangle of their sum and their difference.

V. THE SQUARE ON A SIDE OF A TRIANGLE.

(1) Opposite a right angle, is equivalent to the sum of the squares on the other sides.

(2) Opposite an obtuse angle, is equivalent to the sum of the squares on the other sides plus twice the rectangle of one side by the projection of the other on its line.

(3) Opposite an acute angle, is equivalent to the sum of the squares on the other sides, less twice the rectangle of one side by the projection of the other side on its line; or, in a right triangle, to difference of squares.

VI. THE SUM OF THE SQUARES ON TWO SIDES OF A TRIANGLE.

Is equivalent to twice the sum of the squares on one half the third side and on the median to that side; difference, to two rectangles of the base by its median's projection.

VII. PROPORTIONS BETWEEN SECTS.

(1) Parallels cutting transversals, including triangle case.

(2) Sides of similar polygons are proportional.

(3) In a right triangle with the altitude to the hypotenuse,

(*a*) the altitude is the mean between the sects of the hypotenuse;

(*b*) either leg is the mean between the hypotenuse and the projection of that leg on the hypotenuse.

(4) If two secants cut a circle, the product of the sects from the vertex is the same; a tangent is the mean between the sects of a secant from the same point.

(5) The bisector of an angle of a triangle cuts the opposite side (internally or externally) into sects proportional to the sides including the angle.

(6) Perimeters of regular polygons are proportional to sides, apothems, or radii.

(7) Circumferences are proportional to radii.

VIII. Proportions between Areas.

(1) Areas of triangles having an angle equal are proportional to the product of the sides including the angle; if the triangles are equivalent, the product of the sides in one equals the product in the other, and conversely.

(2) Areas of any similar polygons are proportional to the squares of corresponding sides.

(3) Areas of circles are proportional to the squares of their radii.

IX. The Circumference of a Circle equals $2\pi R$.

X. Altitude.

(1) Of any triangle, is found from III, 4, or by first finding the projection in V, (2) or (3).

(2) In an equilateral triangle, equals half the side times the square root of three.

XI. Projection of a Side of a Triangle, from V, (2), or (3).

XII. Median of a Triangle, from VI.

COLLEGE EXAMINATION QUESTIONS

535. If $ABCDE$ is an inscribed pentagon, and arc DE is 40°, find the number of degrees in the sum of angles A and C.

536. In the triangle ABC, AB equals 7, BC equals 8, and CA equals 5. Find the projection of BC on CA.

537. Construct a square whose area is to that of a given square as 2 is to 5.

538. Construct a circumference equal to the difference of two given circumferences.

539. Construct a triangle, given two sides and the altitude on one of those sides.

540. In a circle of center O, two perpendiculars, OD and OE, are drawn to chords AB and CF, respectively, and angle EDO equals angle DEO. Prove AB equal to CF.

541. Upon the four sides of a square as chords arcs of 90° are constructed within the square. If the diagonal of the square equals $2M$ in., find (*a*) the area bounded by the four arcs; (*b*) the radius of a circle equal to this area.

542. Two equivalent triangles have a common vertex Y, and equal bases AB and CD. If A, B, C, D have fixed positions not in the same line, find the locus of the point Y.

543. The side of an inscribed equilateral triangle is parallel to the side of a regular decagon inscribed in the same circle. Find the number of degrees contained in the intercepted arcs, if the center of the circle lies between the two lines.

544. If the sides of a triangle are 7 and 9 cm., respectively, and the median to the third side equals 7 cm., find the third side.

545. The sides of a polygon are 4, 5, 6, 7, 8 cm. Find the sides of a similar polygon whose area equals four times the area of the given polygon.

546. Two vertices A and B of a triangle have fixed positions. Find the locus of the third vertex C, if angle C is equal to a given angle M, and prove your result.

547. If the base of an isosceles triangle be trisected, and from each point of division a perpendicular be drawn upon the nearest side, prove the perpendiculars equal. State the converse.

548. State two propositions which may be used to prove the equality of arcs. If the lines which join a point in a circumference to the midpoints of two radii are equal, prove that the point bisects the arc subtended by the two radii.

549. Describe briefly a method for proving that the product of two lines is equal to the product of two other lines. If through a point A in a circumference, a tangent AB and a chord AC be drawn, and from C a diameter CD and a perpendicular CE upon AB be drawn, prove that the square of AC equals CE times CD.

550. Construct a circle whose area equals five times the area of a given circle.

551. Prove that the square of a line drawn from the vertex of an isosceles triangle to any point in the base is equal to the square of the leg, diminished by the product of the segments of the base.

552. Determine how many sides the polygon has, the sum of whose interior angles equals the sum of its exterior angles. Explain the method in full.

553. If a fixed arc AB of a circle equals 120°, and a movable arc CD on arc BA equals 60°, find the locus of the intersection of (a) AC and BD; (b) AD and BC.

554. A chord 24 in. long is 9 in. from the center of a circle. Find the length of the tangents drawn from the extremities of the chord and produced till they meet.

555. In the triangle ABC, the angle A is acute and BD is drawn perpendicular to AC. AB equals 10 ft., BC equals 12 ft., and AC equals 14 ft. Find AD.

556. The bases of a trapezoid are 8 in. and 12 in., the area 30 sq. in. Find the length of a line drawn between the legs parallel to the lower base and 2 in. from it.

557. Find the area of the sector of a circle whose radius is 15 in., the angle of the sector being 32°.

558. Show how to construct a circle of given radius tangent to two given circles.

559. Show how to draw a line parallel to the base of an isosceles triangle so that in the trapezoid thus formed the legs shall be equal to the upper base.

560. Prove that the perpendiculars dropped from the midpoints of two sides of a triangle to the third side are equal.

561. Two equal chords produced meet outside the circle. Prove that the secants thus formed are equal.

562. Find the locus of the point at which a given segment of a straight line subtends a given constant angle.

563. A quadrilateral is formed by the diameter AB of a circle, the tangents at A and B, and a third tangent which meets the other tangents at C and D, respectively. Prove that the area is equal to one half the product of the opposite sides AB and CD.

564. A circle of radius 4 has its center at the intersection of the diagonals of a square whose side is 12. Find the length of the circumference of a circle which touches two adjacent sides of the square and also the circle.

565. How many sides has the polygon each of whose interior angles is 175°?

566. If M and N are two lines, construct a line equal to $\sqrt{2\,MN}$.

567. If triangle ABC is equivalent to triangle DEF, A equal to D, $AB = 6$, and $DE = 9$, find the ratio of CA to FD.

568. ABC is a right triangle, C the right angle; $BC = 5$, $CA = 12$, AB is extended to D so $BD = 10$. If CB is extended to E, and DE drawn, how long must BE be in order to make triangle BED equivalent to triangle ABC.

569. If a quadrilateral $ABCD$ is inscribed in a circle, and the diagonals AC and BD meet in E so that $BE = CE$, prove that arc $AB =$ arc CD.

570. Construct a triangle, having given a side and the medians to the other two sides.

571. Each side of a triangle is $2\,n$ cm., and from each vertex as a center with radii equal to n centimeters, circles are drawn. Find the areas bounded (a) by the three arcs that lie without the triangle; (b) by the three arcs that lie within the triangle.

572. Given a square with the side 3 in. long. Find the locus of a point P such that the distance from P to the nearest point on the perimeter of the square is 1 in. Describe the locus accurately.

573. Semicircles are drawn with their centers at the middle points of the sides of an equilateral triangle, forming a triangle less the area of three semicircles. Prove that if the perimeter of this figure is one fifth greater than that of the triangle, its area is about one third less than that of the triangle. If the side of the triangle is 10 in., what is the area of the figure, correct to 1 per cent?

574. On the sides of an equilateral triangle ABC as bases, equal isosceles triangles ABP, ACQ, BCR, are constructed; the first two are exterior to the given triangle, while R is on the same side of BC as A. Prove that $APRQ$ is a rhombus.

575. Find the altitude of an equilateral triangle whose area is $16\sqrt{3}$.

576. If the non-parallel sides of a trapezoid are equal, the angles which they make with the base are equal.

577. Prove that the bisectors of the angles of a quadrilateral form a second quadrilateral of which the opposite angles are supplementary.

578. Through the point M in the base of a triangle parallels to the other two sides are drawn, forming a parallelogram. Find the locus of the center of the parallelogram as M moves along the base of the triangle.

579. Find to one place of decimals the area of a six-pointed star formed by joining the alternate vertices of a regular hexagon inscribed in a circle of radius 3 in.

580. The three angles of a triangle are 48°, 82°, and 50°. Find the three angles formed by the bisectors of the angles of the triangle. Verify by using the theorem involving the sum of the angles about a point in a plane.

581. A chord 1 ft. long is 4 in. from the center of a circle. How far from the center is a chord 9 in. long?

582. A circle has an area of 80 sq. ft. Find the length of an arc of 80°.

583. Prove that if a median of a triangle is equal to half the side to which it is drawn, the triangle is a right triangle.

584. Prove that if AB is a diameter of a circle, and BC a tangent, and AC meets the circumference at D, the diameter is a mean proportional between AC and AD.

585. Given three lines a, b, c. Construct a line x so that $a:b::c:x$.

586. Two parallelograms are equal if two sides and the included angle of one are equal to two sides and the included angle of the other.

587. The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

588. Three equal circles are described each tangent to the other two. If the common radius is R, find the area contained between the circles.

589. The radius of a circle is 10 ft.; the area of a sector of that circle is 130 sq. ft. What is its arc in degrees?

590. Two sides and a diagonal of a parallelogram are 7, 9, and 8, respectively. Find the length of the other diagonal.

591. One of two secants meeting without a circle is 12.5 in., and its external segment is 4 in.; the other secant is divided into two equal parts by the circumference. Find the length of the second secant.

592. $ABCD$ is any parallelogram, and E any point within it. Prove that the sum of the triangles EAD and EBC equals half the area of the parallelogram.

593. Three consecutive angles of an inscribed quadrilateral subtend arcs of 70°, 85°, and 98°, respectively. Find each angle of the quadrilateral and the angle between the diagonals.

594. The diameters of two concentric circles are 16 and 40 ft., respectively. Find the length of a chord of the larger which is tangent to the smaller.

595. The area of a rhombus is 96 sq. ft. and its side is 10 ft. Find the lengths of its diagonals.

596. Show how to inscribe a circle in a given sector.

597. Show how to divide a triangle into two equivalent parts by a line parallel to one of its sides.

598. Prove that the tangents to two intersecting circles from any point in their common chord produced are equal.

599. Two tangents drawn from the same external point to a circle form an angle of 64°. Find the number of degrees in each of the arcs intercepted by these tangents.

600. Find the radius of a circle whose circumference numerically equals its area.

601. From a point 12 in. from the center of a circle 16 in. in diameter, two tangents are drawn to the circumference. Find the length of the chord joining the points of contact.

602. Prove that if a circle is circumscribed about an isosceles triangle, the tangents through the vertices form another isosceles triangle.

603. Two circles are tangent externally and through the point of contact two straight lines are drawn terminating in the circumferences. Prove that the corresponding segments of the lines are proportional.

604. Show how to draw a line terminating in the arms of an angle, which shall be equal to one given line and parallel to another.

605. Two circles are tangent internally. If chords of the larger circle are drawn through the point of contact, prove that they are divided proportionally by the smaller circle.

606. Prove that if one acute angle of a triangle is double another, the triangle can be divided into two isosceles triangles by a straight line through the vertex of the third angle.

607. Given a regular hexagon each side of which is 6 in. With three of the alternate vertices as centers, arcs of circles are drawn passing through the center of the polygon. Find the area of the three loops thus formed.

608. A point A is 4 ft. from the circumference of a circle; the length of a tangent from A to the circle is 10 ft. Find the diameter.

609. The bases of a trapezoid are respectively 29 ft. and 37 ft. and its area is 247.5 sq. ft. Find its altitude.

610. Show how to divide a given rectangle into four equivalent parts by lines drawn from one of the vertices of the rectangle.

611. Given a straight line and two points on the same side of that line and at unequal distances from it. Construct a circumference passing through the two points and having its center in the given line.

612. Prove that the area of a square inscribed in a given circle is twice the square of the radius of the circle.

613. ABC is a triangle inscribed in a circle with center O. Take D the middle point of the arc BC and draw OD and AD. Prove that the angle ADO equals half the difference of the angles B and C.

614. If a line is drawn from the vertex A of triangle ABC to any point D of the opposite side, and any point O on AD is joined to B and C, prove $\frac{\triangle ABC}{\triangle OBC} = \frac{AD}{OD}$.

615. From a point at a distance of 10 in. from the center of a circle of radius 5 in., two tangents are drawn. Compute the area bounded by the tangents and their included arc.

616. Construct a triangle having given the midpoints of its three sides.

617. The legs of a right triangle are 6 and 8, respectively. Find their projections on the hypotenuse.

618. Compute the area of a square inscribed in a circle whose perimeter is 63 ft.

619. The legs of a right triangle are respectively 15 ft. and 8 ft. Find the length of each segment made on the hypotenuse by the bisector of the right angle.

620. Find the product of the segments of any chord drawn through a point 12 in. from the center of a circle whose diameter is 18 in.

621. Find the area of an equilateral triangle inscribed in a circle of radius 4 in.

622. Show how to construct a square which shall be one third of a given square.

623. Prove that if in a right triangle one angle is 30°, the hypotenuse is double the shorter leg.

624. Prove that two chords drawn perpendicular to a third chord at its extremities are equal.

625. Draw a square $ABCD$. On the diagonal AC take the point E so that $AE = AB$, and draw through E a perpendicular to AE, cutting BC in F. Prove that $BF = FE = EC$.

626. Take five points on the circumference of a circle (A, B, C, D, E), in the order named. Let the middle point of the arc ABC be F, and the middle point of the arc AED be G. Let the chord FG cut the chord AC in H and the chord AD in K. Prove that AH is equal to AK.

627. Prove that if in a right triangle the hypotenuse is double the shorter leg, one acute angle is double the other.

628. Show how to construct a square equivalent to a given parallelogram. Prove the correctness of your method.

629. Prove that if one leg of a right triangle is the diameter of a circle, the tangent at the point where the circumference cuts the hypotenuse bisects the other leg.

630. The apothem of a regular hexagon is $6\sqrt{3}$. Find the area of the hexagon and the area of the inscribed circle.

631. Prove that if a tangent be drawn from one end of a diameter meeting a secant from the other end, the product of the secant and the *internal* segment will be the same for all directions of the secant.

632. Prove that if two circles are tangent internally at A and a straight line intersects the two circles in B, C, D, and E, then angle $BAC = DAE$.

633. If in triangle ABC the line from C to D on AB bisects angle C, and $AB = 2.09$, $BC = 3.14$, $CA = 4$, compute values of DB and DA, respectively.

634. To measure the height of a church spire, a rod 10 ft. long is planted vertically at position A, then at position B. In each case the observer takes such a position that the top of the spire, the top of the rod, and his eye are all in line when he stands erect. He measures the distance each time from his position to the rod, and obtains the measurements 4 ft. and 8.3 ft. He also measures the distance between his two positions, finding it to be 138 ft. If the observer's eye is 5 ft. above the ground, what is the height of the spire? Explain your solution.

635. What is the locus of the midpoint of one leg of a right triangle whose hypotenuse is fixed? Prove the correctness of your answer.

636. In a quadrilateral $ABCD$ the lengths of AB and BC are equal and angle A is greater than angle C. Which is the longer side, AD or CD? Give the reason for your answer.

637. Prove that in a convex quadrilateral the angle between the bisectors of two adjacent angles is one half the sum of the other two angles.

638. An arc of a certain circle is 100 ft. long, and subtends an angle of 25° at the center. Compute the radius of the circle correct to three significant figures.

639. Three successive vertices of a regular octagon are A, B, C, respectively. If the length AB is a, compute the length AC.

640. The areas of similar segments of circles are proportional to the squares of their radii.

641. Given a circle whose radius is 16. Find the perimeter and the area of the regular inscribed octagon.

642. Two circles intersect at points A and B. Through A a variable secant is drawn, cutting the circles in C and D. Prove that the angle CBD is constant for all positions of the secant.

643. Let A and B be two fixed points on the circumference of a given circle and P and Q the extremities of a variable diameter of the same circle. Find the locus of the point of intersection of the straight lines AP and BQ.

644. Prove that in any right triangle the line drawn from the right angle to the middle of the hypotenuse is equal to one half the hypotenuse.

645. The area of a regular decagon is 108 sq. in. Find the radius of the circumscribed circle.

646. Two secants are drawn from the same point to the same circle. The external segment of the first is 5 in. and its internal segment is 19 in. The internal segment of the other secant is 7 in. Find the length of the second secant.

647. On the diameter AB of a circle mark a point P. Through P draw the chord CPD at right angles to AB. Prove that if AP, BP, CP, and DP be taken as diameters of circles, the sum of the areas of the four circles is equal to the area of the original circle.

648. To construct a rectangle, having given the perimeter and the diagonal.

649. Find how far from the base of a triangle of altitude a lines parallel to the base must be drawn to divide the area of the triangle into three equal parts.

650. Prove that two triangles are similar if the sides of one are respectively parallel to the sides of the other.

651. Derive the numerical value of π.

652. Of all triangles having the same base and equal areas, the isosceles triangle has the minimum perimeter.

653. How high is a tree which casts a shadow 70 ft. long, when a man 6 ft. high casts a shadow $8\frac{3}{4}$ ft. long?

654. Construct a triangle, given the base, vertical angle, and median drawn to the base.

655. State six propositions concerning parallel lines and prove any one of them.

656. An interior angle of an equiangular polygon is 150°. Find the number of sides.

657. Three cylindrical barrels, diameter of each being 20 in., are placed in a pile with axes horizontal so that each just touches the other two. Find the height of the pile, and the length of the shortest rope to go over the pile and touch the floor on each side.

658. AB is the diameter of a circle of radius 2 in., and AC is a chord such that BAC is 30°. Find area and perimeter of $BACB$ correct to two decimal places.

659. If from a fixed point D, within a triangle ABC, lines are drawn to all points in the perimeter of the triangle, what is the locus of the middle points of those lines?

660. Show that if the radius of a circle is a, the side of the regular inscribed decagon is $\frac{a}{2}(\sqrt{5}-1)$ and the side of the regular inscribed pentagon is $\frac{a}{2}\sqrt{10-2\sqrt{5}}$.

661. The base of a triangle is 32 ft. and its height is 20 ft. What is the area of the triangle formed by drawing a line parallel to the base 5 ft. from the vertex?

662. The sides of a triangle are 5, 12, 13. Find the segments into which each side is divided by the bisector of the opposite angle.

663. The sides of a triangle are a, b, c, and the area is k. What is the radius of the inscribed circle?

664. A and B are fixed points, AC is drawn in any direction, and BP is drawn perpendicular to AC, meeting it at P. What is the locus of P?

665. Upon a line about $1\frac{1}{2}$ in. long construct a segment to contain an angle of 60°.

666. Three circles of radius a touch each other, and another circle is circumscribed about them. Find its radius, circumference, and area.

667. AB is a fixed line, angle $ACB = 45°$. Construct the locus of C.

668. Two sides of a triangle are a and b, the included angle 135°. What is the square of the third side c?

669. The lengths of the circumferences of two concentric circles differ by 6 in. Compute the width of the ring to three significant figures.

670. Prove the area of the triangle formed by joining the middle point of one of the nonparallel sides of a trapezoid to the extremities of the opposite side equals one half the trapezoid.

671. The three sides of a triangle are 4 ft., 13 ft., and 15 ft. long. Show that the altitude upon the side of length 15 is 3.2.

672. Through the vertex A of the parallelogram $ABCD$ draw a secant. Let this line cut diagonal BD in E, and the sides BC, CD (or these sides produced) in F and G, respectively. Prove that AE is a mean proportional between EF and EG.

673. Let ABC be an equilateral triangle, and on the sides AB, BC, CA, lay off AD, BE, CF, each equal to one third AB, and join the points D, E, F, with one another. Prove that the triangle DEF is equilateral, and that its sides are respectively perpendicular to the sides of the given triangle.

674. On the circumference of a circle take two points subtending a right angle at the center, and a third point on the arc between these two. Prove that the perimeter of the triangle formed by the tangents at these three points is equal to the diameter of the circle.

675. An indefinite straight line moves in such a way that it always passes through at least one vertex of a given square, but never crosses the square. What is the locus of the foot of the perpendicular dropped on the moving line from the center of the square? Describe the locus accurately and prove the correctness of your answer.

676. Prove the correctness of the following construction for bisecting an angle ABC; upon AB produced beyond B take BD equal to BC and draw a line through B parallel to DC.

677. Show how to construct a chord through a given point A within a circle, so that the extremities of the chord shall be equidistant from another point B.

678. A rod of length a is free to move within a semicircular area of radius a. Describe accurately the boundary of the region within which the middle point of the rod will always be found.

679. A roadway 60 ft. wide is cut through the middle of a circular field 120 ft. in diameter. Compute the area of the remainder of the field correct to 1 per cent of its value.

680. The radii of two circles are 1 in. and $\sqrt{3}$ in., respectively, and the distance between their centers is 2 in. Compute their common area to three significant figures.

681. Determine a point P without a given circle so that the sum of the tangents from P to the circle shall be equal to the distance from P to the farthest point of the circle.

682. The image of a point in a mirror is, apparently, as far behind the mirror as the point itself is in front. If a mirror revolves about a vertical axis, what will be the locus of the apparent image of a fixed point one foot from the axis?

683. The hypotenuse of a right triangle is 10 in. long and one of the acute angles is 30°. Compute the lengths of the segments into which the short side is divided by the bisector of the opposite angle.

684. A chord BC of a given circle is drawn, and a point A moves on the longer arc BC. Draw triangle ABC and find the locus of the center of a circle inscribed in this triangle.

685. Three equal circular plates are so placed that each touches the other two, and a string is tied tightly around them. If the length of the string is 10 ft., find the radius of the circles correct to three figures.

APPENDIX

339. Contraposite Law. This law was stated in § 8, but no explanation of why it was true was attempted. The following explanation is simple, and shows very plainly that the law holds for all statements.

Given. If A, then B.

To prove. If not B, then not A.

Proof. I. If not B, then either (1) A,
or (2) not A (all possibilities).

II. But, if A, then B (given).
∴ using (1), if not B, then A, then B.

III. This is impossible, for it contradicts itself.
∴ If not B, then not A
(only other possibility).

(This proof depends upon the "Law of Excluded Mean," but for the purpose of this proof it is not necessary to discuss that Law.)

340. Law of Converse. Stated in § 10.

Given. If A, then X,
If B, then Y,
If C, then Z, etc.
Where A, B, C, . . . cover all possibilities, and no two of the conclusions X, Y, Z, . . . can be true at once.

To prove. If X, then A;
If Y, then B;
If Z, then C, etc.

Proof. I. If X, then not Y, Z, or any other conclusion (no two can be true at once, by the given).

II. $\therefore$ not B, not C, not any condition but A (contraposite).

III. $\therefore$ If X, then A (only remaining possibility).

341. Proofs of the Obverse and the Converse of a Single Statement. (1) When it is necessary to prove the obverse of a single statement, take both conditions, and show that they cannot give the same conclusion.

If A, then B. Take "not A," and show that it cannot also give B.

See § 122; in this case A represents equal angles, B represents parallel lines. Then "not A" represents unequal angles, and the reason it cannot also give B, that is, parallel lines, is that there can be but one parallel through a point.

(2) To prove the converse of a single statement, when that is necessary, show that the condition and the conclusion give one and the same thing.

If A, then B. Show that B gives the same as A.

In § 124, if it is to be proved as the converse of § 121, the proof would be as follows, the work being much abbreviated.

If A (that is, the angles are equal), then B (that is, the lines are parallel).

If B (that is, the lines are parallel). But there is but one parallel through the point in question, therefore this figure is identical with the other, and so the angles are equal.

Note the application of this method to § 144.

In both these methods the proof follows because of some element of exclusion, that is, some statement that there can be but one of a certain thing.

342. General Condition. In many of the theorems of Geometry the hypothesis contains two kinds of conditions, one of which might be called the *general condition*, for it states the kind of figure about which the statement is to be made, while the other might be called the *special condition*, for it is that on account of which the conclusion follows. For example, in Th. I, the general condition is the given triangles; the special condition, the fact that they have two sides and the included angle equal. If the converse, obverse, or contraposite is to be stated, *the general condition should be left just as in the original statement, the special condition and the conclusion being used.*

The subject could be made very complicated by going into all the different ways of using the two conditions, but the statements obtained by following the direction given above will be sufficient for all practical purposes.

343. Axioms. The axioms, as used in this book, are not entirely independent; for example, the substitution axiom could be derived from the equality axioms, as far as the equal cases are concerned. The position of the author in this respect has been defined in the preface. The following axiom can be derived from the axiom of intersection, but the work is not worth while, although the axiom is needed for certain work in the theorems and exercises.

Diagonal Intersection Axiom. *The diagonals of any convex quadrilateral intersect each other at a point within the quadrilateral.*

344. Symmetry.

(1) *With respect to a center.* Two points are said to be symmetrical with regard to a point (or center) when they are on opposite sides of the point on the same straight line, and at the same distance from that point. Two

figures are said to be symmetrical with regard to a center when all their corresponding points are respectively symmetrical with regard to the center.

(2) *With respect to an axis.* Two points are said to be symmetrical with regard to an *axis* when they are on opposite sides of a line (or axis) on a perpendicular to the axis, and equidistant from the axis. Two figures are said to be symmetrical with regard to an axis when any two corresponding points are symmetrical with regard to that axis.

The placing of figures in symmetrical positions, as in Bk. I, Th. VII (§ 105), is quite common, as is the use of symmetrical points. Note also the figure of § 100, and exercise *120*.

345. Positive and Negative Sects and Angles. There are many theorems in Geometry where different cases of the theorem require that sects or angles be added in one case, subtracted in a second, and in the third one of the things added or subtracted is zero. If these are arranged in one general statement, it can be shown that the sects, or angles, are all positive in the addition case, that one pair has become zero in the zero case, and then negative in the subtraction case. Examples of this can be found in the proof of Th. VII, Bk. I (§ 105), in §§ 207, 214, and 215, and in §§ 240 and 244.

346. Distance. Where the word "distance" is used, it always means the shortest possible distance.

The distance between two points means the straight-line distance. That this is the shortest line has been partly proved in § 109, where the straight line between two points is shown to be less than any broken line between the points. That it is less than any curved line between the points has not been shown, but it can be proved quite easily by showing that the shortest line between two points must pass through *any* point on the straight line between those points, and therefore through *every* point on the straight line.

The distance from a point to a line is the perpendicular from the point to the line (§ 113).

The distance from a point to a circumference is the sect from the point to the circumference on the line through the center (§ 174).

347. Limits. There is some doubt as to the perfect accuracy of any of the proofs for the first limit theorem; the following proof is probably as free from objections as any.

I. *If two variables approaching limits are equal for all values, their limits are also equal.*

Given. $v = v'$; $v \doteq L$, $v' \doteq L$.

To prove. $L = L'$.

Proof. I. $v = L - x$.

$v' = L' - x'$, where x and x' are quantities, either positive or negative, which can become indefinitely small, but cannot equal zero.

II. $v = v'$ (given), $\therefore L - x = L' - x'$ (eq. same), and $L - L' = x - x'$ (eq. +, − eq.).

III. L, L' and $L - L'$ are constants; x and x' can each be made less than any constant except zero. If $L - L'$ is not zero, x and x' can each become less than $\frac{1}{2}(L - L')$, and then $x - x'$ would be less than $L - L'$, even though the signs were such that the values of x and x' were added.

IV. But this is impossible, since $L - L' = x - x'$.
$\therefore L - L' = 0$.

II. (1) *If a variable approaches the limit zero, its quotient by a constant, and its product by a constant (other than zero), approaches zero.*

Given. $v \doteq 0$; c.

To prove. $\frac{v}{c} \doteq 0$; $vc \doteq 0$.

Proof. I. Let k be any constant quantity, however small. Then ck is a constant quantity, and v can become less than ck.

II. But if $v < ck$, then $\frac{v}{c} < k$ (uneq. ÷ eq.).

III. And $\frac{v}{c} \neq 0$, since $v \neq 0$ (def.).

IV. Since $\frac{v}{c}$ is less than k, a constant quantity, however small, but is not zero, $\therefore \frac{v}{c} \doteq 0$ (def.).

Similarly $vc \doteq 0$.

(2) *If a variable approaches any limit, its quotient by a constant, and its product by a constant (other than zero), will approach the quotient of its limit by the constant, and the product of its limit by the constant.*

Given. $v \doteq L$; c.

To prove. $\frac{v}{c} \doteq \frac{L}{c}$; $vc \doteq Lc$.

Proof. I. If $v \doteq L$, then $L - v \doteq 0$ (def.).

II. $\therefore \frac{(L - v)}{c} \doteq 0$ (by II (1)), or, $\frac{L}{c} - \frac{v}{c} \doteq 0$.

III. $\therefore \frac{v}{c} \doteq \frac{L}{c}$ (def.).

Similarly $vc \doteq Lc$.

III. *If two variables are proportional to two constants, their limits are proportional to the same constants.*

Given. $\frac{v}{c} = \frac{v'}{c'}$; $v \doteq L$, $v' \doteq L'$.

To prove. $\frac{L}{c} = \frac{L'}{c'}$.

Proof. I. Since $v \doteq L$, $\frac{v}{c} \doteq \frac{L}{c}$ (II).

II. Since $v' \doteq L'$, $\frac{v'}{c'} \doteq \frac{L'}{c'}$ (II).

III. But $\frac{v}{c} = \frac{v'}{c'}$. $\therefore \frac{L}{c} = \frac{L'}{c'}$ (I).

348. Incommensurable Case. The incommensurable case of Th. V, Bk. II, will be taken as an example of the method to be applied to all theorems having the two cases.

Let the central angles AOB and COD in circle O have no common divisor. Suppose an exact divisor of $\angle AOB$ to be applied to $\angle COD$ as often as possible, leaving a remainder $\angle XOD$, which is therefore less than the divisor used.

Then $\angle AOB$ and $\angle COX$ have a common divisor, so $\frac{\angle AOB}{\angle COX} = \frac{\widehat{AB}}{\widehat{CX}}$ by the commensurable case.

But, if the divisor of $\angle AOB$ is taken smaller and smaller, that is, is made to approach the limit zero, the remainder $\angle XOD$, being still smaller, will also approach the limit zero. Therefore $\angle COX$ will approach the limit $\angle COD$. Also its arc, $\widehat{CX}$, will approach the limit $\widehat{CD}$.

But, since the variables $\angle COX$ and $\widehat{CX}$ are proportional to the constants $\angle AOB$ and $\widehat{AB}$, their limits are also proportional to those constants; that is, $\frac{\angle AOB}{\angle COD} = \frac{\widehat{AB}}{\widehat{CD}}$.

This method of proof will apply to Th. II, Bk. III, to Th. I, Bk. IV, and in fact to all proofs where the method of a common divisor is used.

349. Similar Figures. Similar figures have already been defined (§ 282) as those which have their corresponding angles equal, and their corresponding sides proportional. This, of course, applies only to polygons. The following definition is sometimes given, and while it is not as convenient for use in Plane Geometry, it has the merit of applying to all kinds of figures, including those of Solid Geometry.

Two figures that may be placed in a pencil (or *sheaf*, in Solid Geometry) of lines, so that all pairs of corresponding points of the figures cut the respective lines of that pencil in the same ratio, are similar.

These two definitions are identical in result, as far as polygons are concerned, as may be shown by proving the two following statements:

1. *If two polygons that are mutually equiangular and have their corresponding sides proportional are placed with one pair of corresponding sides parallel, the polygons lying on the same side of those lines, the lines joining their corresponding vertices will form a pencil which is cut proportionally by the vertices of the polygons.*

2. *If two polygons lie in a pencil of lines, and their vertices cut the lines proportionally, the polygons are mutually equiangular, and their corresponding sides are proportional.*

350. The Evaluation of Pi. The ratio of the circumference of a circle to its diameter is obtained by the use of the ratios of the perimeters of regular inscribed and circumscribed polygons of a large number of sides to the diameter. The circumference is longer than the perimeter of any inscribed polygon, and shorter than the perimeter of any circumscribed polygon, so that if the lengths of the perimeters of two regular polygons of the same number of sides, one inscribed, the other circumscribed, can be obtained in terms of the diameter, the length of the circumference will lie between these values, and an approximate value can therefore be obtained.

The value $\frac{C}{D}$, or π, cannot be obtained exactly, for it has been proved to be an incommensurable number. It is evident, however, that the value obtained will be more exact as the polygons used have a larger number of sides.

The approximation is started by finding the perimeters of some regular polygon, inscribed and circumscribed; squares or hexagons are the easiest to use. The perimeters of polygons of twice the number of sides are then worked out, and the doubling is continued until the value is found to the required degree of accuracy. The value has been carried to over 700 decimal places, but for ordinary purposes the value 3.14159^{+} is sufficiently accurate. 3.1416 is also much used, and $3\frac{1}{7}$ is a fair approximation for rough work.

The following theorems give the material for the numerical calculation:

I. *The perimeter of a regular inscribed hexagon equals* $3D$*; the perimeter of a regular circumscribed hexagon equals* $2D\sqrt{3}$.

II. *If the perimeters of regular inscribed and circumscribed polygons of any number of sides are known, the perimeter of the regular circumscribed polygon of double the number of sides equals twice their product divided by their sum.*

III. *If the perimeters of the regular inscribed poly-*

gon of any number of sides, and the regular circumscribed polygon of double that number of sides, are known, the perimeter of the regular inscribed polygon having double the first number of sides equals the square root of their product.

Applying the formula in II to the regular hexagons, calling the perimeter of a regular circumscribed dodecagon P:

$$P = \frac{2(3\,D)(2\,D\sqrt{3})}{3\,D + 2\,D\sqrt{3}} = \frac{12\,D\sqrt{3}}{3 + 2\sqrt{3}} = 12\,D(2 - \sqrt{3}),$$

or, $P = (3.21539^{+})D$.

Applying the formula in III to this value and the perimeter of the regular inscribed hexagon, and calling the perimeter of the regular inscribed dodecagon p:

$$p = \sqrt{3\,D \times 12\,D(2 - \sqrt{3})} = 6\,D\sqrt{2 - \sqrt{3}} = (3.10583^{-})D.$$

Applying these methods to the dodecagons, the perimeters of circumscribed and inscribed regular polygons of 24 sides are, respectively, $(3.15966^{-})D$ and $(3.13263^{-})D$.

It is already evident that the circumference of the circle, since its value is between those found, must be $(3.1^{+})D$, and a continuance of the work will determine additional figures.